Normal Impact of a Viscoelastic Spherical Shell against a Rigid Plate

Y. ROSSIKHIN, M. SHITIKOVA
Voronezh State University of Architecture and Civil Engineering
Research Center on Dynamics of Structures
20-letiya Oktyabrya Str. 84, Voronezh 394006
RUSSIAN FEDERATION
yar@vgasu.vrn.ru, mvs@vgasu.vrn.ru

Duong Tuan MANH
Voronezh State University of Architecture and Civil Engineering
on leave from Hanoi University of Architecture, VIETNAM
iam.mr.manh@gmail.com

Abstract: In the present paper, the normal impact of a viscoelastic spherical shell upon a rigid plate is investigated using the wave theory of impact. The model developed here suggests that after the moment of impact quasi-longitudinal and quasi-transverse shock waves are generated, which then propagate along the spherical shell. The solution behind the wave fronts is constructed with the help of the theory of discontinuities. Since the local bearing of the material of the impactor is taken into account, then the solution in the contact domain is found via the modified Hertz contact theory involving the operator representation of viscoelastic analogs of Young's modulus and Poisson's ratio.

Key–Words: Fractional derivative standard linear solid model, Impact, Viscoelastic spherical shell

1 Introduction

Nowadays fractional calculus is widely used in different fields of science and technology, including various dynamic problems of mechanics of solids and structures [1], and the problems of impact interaction among them [2].

Thus, recently Rossikhin et al. [3] investigated the collision of two viscoelastic shells, viscoelastic features of which are described by the standard linear solid model with conventional integer derivatives. During the impact process there occurs decrosslinking within the domain of the contact of the colliding bodies, resulting in more freely displacements of molecules with respect to each other, and finally in the decrease of the shells' material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the materials of the colliding spherical shells within the contact domain by the standard linear solid model involving fractional derivatives, since variation in the fractional parameter (the order of the fractional derivative) enables one to control the viscosity of the shells' material. That is why the fractional parameter could be considered as the structural parameter.

In the present paper, we will consider a special but very important for engineering practice case when a viscoelastic spherical shell impacts a rigid plate.

2 Problem Formulation

Let us consider the problem on a normal impact of a viscoelastic spherical shell with the initial velocity V_0 against a rigid plate (Fig. 1), when the viscoelastic features of the impactor are described by the standard linear solid model with conventional derivatives of integer order.

For this purpose we will proceed from equations of motion of two colliding viscoelastic spherical shells derived recently in [3], wherein we tend the radius and Young's modulus of the second shell to infinity. As a result we obtain the following equation of motion of the contact domain

$$\rho\pi a^2 h\dot{v}_z = 2\pi a h\sigma_{rz}|_{r=a} + F_{\rm cont} \tag{1}$$

under the action of the transverse force $2\pi ah\widetilde{\sigma_{rz}}|_{r=a}$ and $F_{\rm cont}$ is the contact force which is defined via the generalized Hertzian contact law

$$F_{\rm cont} = \tilde{k}\alpha^{3/2}, \tag{2}$$

where α is the local bearing of the impactor's material (Fig. 2), $\tilde{k}$ is the operator involving the geometry, i.e. spherical shell radius R, and viscoelastic features of the impactor defined by the time-dependent functions $\tilde{E}$ and $\tilde{\nu}$

$$\tilde{k} = \frac{4}{3}\,\frac{\sqrt{R}\tilde{E}}{1-\tilde{\nu}^2}, \tag{3}$$

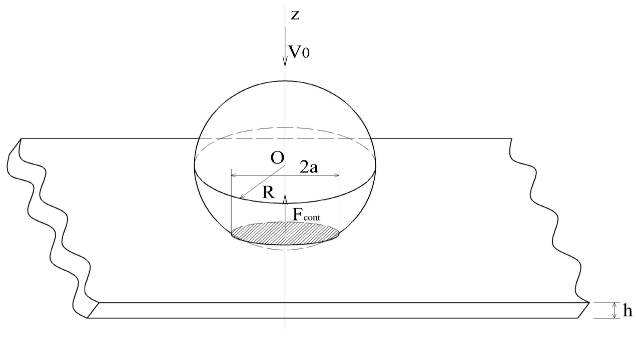

Figure 1: Scheme of the normal impact of a spherical shell against an infinite plate

ρ and h are the density and thickness of the shell, respectively, a is the radius of the contact domain (Fig. 1), and an overdot denotes the time-derivative.

The following equation

$$V_0 - v_z|_{r=a} = \dot{\alpha} \tag{4}$$

should be added to equations (1) and (2).

In [3] it has been shown that considering $v_r|_{r=a} = \dot{a}$, the value $\sigma_{rz}|_{r=a}$ could be calculated in the following form according to the dynamic condition of compatibility:

$$\sigma_{rz}|_{r=a} = \rho\,(G_1 - G_2)\,\frac{(a^2)^{\cdot}}{2R} - \rho\left(G_1\frac{a^2}{R^2} + G_2\right)v_z|_{r=a}, \tag{5}$$

where G_1 and G_2 are the velocities of the quasi-longitudinal and quasi-transverse waves (surfaces of strong discontinuity), respectively, which are generated at the moment of impact at the point of tangency (or the point of contact) of the impactor with the target, which then propagate in the form of diverging circles along spherical surface, and are defined as

$$G_1 = \sqrt{\frac{E_\infty}{\rho(1-\nu_\infty^2)}}, \tag{6}$$

$$G_2 = \sqrt{\frac{\mu_\infty}{\rho}}, \tag{7}$$

where E_∞, μ_∞ and ν_∞ are non-relaxed elastic and shear moduli and Poisson's ratios, respectively.

Considering that $a/R \ll 1$, equation (5) is reduced to

$$\sigma_{rz}|_{r=a} = -\rho G_2 v_z|_{r=a}, \tag{8}$$

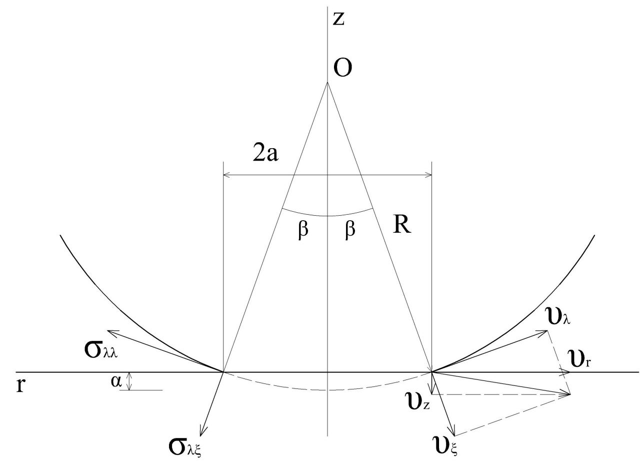

Figure 2: Scheme of velocities and stresses in the shell's element on the boundary of the contact domain [3]

Now substituting (2) and (8) in (1) and considering that $a^2 = R\alpha$ yield

$$\rho\pi R\alpha h\dot{v}_z|_{r=a} = -2\pi(R\alpha)^{1/2}h\rho G_2 v_z|_{r=a} + \tilde{k}\alpha^{3/2}. \tag{9}$$

In order to solve equation (9), we should define the operator $\tilde{k}$, resulting in decoding the operator $\widetilde{E}/(1-\widetilde{\nu}^2)$.

For the majority of viscoelastic materials, the bulk modulus K remains constant during the process of mechanical loading of this material [4], resulting in [2]

$$\frac{\widetilde{E}_1}{1-2\widetilde{\nu}} = \frac{E_\infty}{1-2\nu_\infty}. \tag{10}$$

Recently it has been proposed in [3] that during the impact process there could occur decrosslinking within the domain of the contact between the impactor and target, resulting in more freely displacements of molecules with respect to each other, and finally in the decrease of the shell's material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the material of the impacting spherical shell within the contact domain by the standard linear solid model involving fractional derivatives

$$\sigma + \tau_\varepsilon^\gamma D^\gamma \sigma = E_0\left(\varepsilon + \tau_\sigma^\gamma D^\gamma \varepsilon\right), \tag{11}$$

where σ is the stress, ε is the strain, E_0 is the relaxed modulus, τ_ε and τ_σ are the relaxation and creep times, respectively,

$$D^\gamma x(t) = \frac{d}{dt}\int_0^t \frac{(t-t')^{-\gamma}}{\Gamma(1-\gamma)}\,x(t')dt' \tag{12}$$

is the Riemann-Liouville fractional derivative, $\Gamma(1-\gamma)$ is the Gamma-function, γ $(0<\gamma\le 1)$ is the fractional parameter, and $x(t)$ is a certain function.

Utilizing the model (11), it could be found [2] that

$$\frac{\widetilde{E}}{1-\widetilde{\nu}^2}=\frac{E_\infty}{1-\nu_\infty^2}\left[1-m_1\ni_\gamma^*\left(t_1^\gamma\right)-m_2\ni_\gamma^*\left(t_2^\gamma\right)\right], \tag{13}$$

where $\ni_\gamma^*\left(t_i^\gamma\right)$ $(i=1,2)$ is the dimensionless Rabotnov operator [2]

$$\ni_\gamma^*\left(t_i^\gamma\right)=\frac{1}{1+t_i^\gamma D^\gamma}, \tag{14}$$

and

$$t_1^\gamma=\frac{2(1+\nu_\infty)\tau_\varepsilon^\gamma}{2(1+\nu_\infty)+\nu_\varepsilon(1-2\nu_\infty)},$$
$$t_2^\gamma=\frac{2(1-\nu_\infty)\tau_\varepsilon^\gamma}{2(1-\nu_\infty)-\nu_\varepsilon(1-2\nu_\infty)},$$
$$m_1=\frac{3}{2}\,\frac{(1-\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty)+(1-2\nu_\infty)\nu_\varepsilon},$$
$$m_2=\frac{1}{2}\,\frac{(1+\nu_\infty)\nu_\varepsilon}{2(1-\nu_\infty)-(1-2\nu_\infty)\nu_\varepsilon},$$
$$\nu_\varepsilon=\frac{E_\infty-E_0}{E_\infty}.$$

Equation (9) with due account for (4) and (13), as well as the initial conditions

$$\alpha|_{t=0}=0,\quad \dot\alpha|_{t=0}=V_0, \tag{15}$$

is reduced to

$$\ddot\alpha+\text{æ}\left[\alpha^{1/2}(t)-\Delta_\gamma\alpha^{-1}\int_0^t(t-t')^{\gamma-1}\times\alpha^{3/2}(t')dt'\right]=0, \tag{16}$$

where

$$\text{æ}=\frac{4E_\infty}{3\pi\sqrt{R}\rho h(1-\nu_\infty^2)},\quad \Delta_\gamma=\frac{1}{\Gamma(\gamma)}\sum_{j=1}^2\frac{m_j}{t_j^\gamma}.$$

3 Approximate Solutions

If we consider

$$\alpha\approx V_0 t \tag{17}$$

as a first approximation, then Eq. (16) with due account for

$$\int_0^t\left(t-t'\right)^{\gamma-1}t'^{3/2}dt'=\frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)t^{3/2+\gamma} \tag{18}$$

takes the form

$$\ddot\alpha=-\text{æ}V_0^{1/2}\left[t^{1/2}-\Delta_\gamma\frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)t^{1/2+\gamma}\right]. \tag{19}$$

Integrating (19) yields

$$\dot\alpha = V_0-\frac{2}{3}\text{æ}V_0^{1/2}t^{3/2}+\text{æ}V_0^{1/2}\Delta_\gamma\frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)\frac{t^{3/2+\gamma}}{3/2+\gamma}, \tag{20}$$

and

$$\alpha = V_0t-\frac{4}{15}\text{æ}V_0^{1/2}t^{5/2}+\text{æ}V_0^{1/2}\Delta_\gamma\frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)\frac{t^{5/2+\gamma}}{(3/2+\gamma)(5/2+\gamma)} \tag{21}$$

3.1 The case $\gamma=0$

In a particular case, when $\gamma=0$, and therefore

$$\sum_{j=1}^2 m_j=0,$$

relationships (20) and (21) take the form

$$\alpha=V_0\left(1-\frac{2}{3}\text{æ}V_0^{-1/2}t^{3/2}\right), \tag{22}$$

$$\alpha=V_0t\left(1-\frac{4}{15}\text{æ}V_0^{-1/2}t^{3/2}\right), \tag{23}$$

from which the contact duration $t_{\rm cont}^{(0)}$ and the time $t_{\rm max}^{(0)}$ at which the maximal local indentation $\alpha_{\rm max}^{(0)}$ takes place could be found

$$t_{\rm cont}^{(0)}\approx\left(\frac{15}{4}\frac{V_0^{1/2}}{\text{æ}}\right)^{2/3}, \tag{24}$$

$$t_{\rm max}^{(0)}\approx\left(\frac{3}{2}\frac{V_0^{1/2}}{\text{æ}}\right)^{2/3}, \tag{25}$$

$$\alpha_{\rm max}^{(0)}\approx\frac{3}{5}V_0t_{\rm max}^0. \tag{26}$$

3.2 The case $\gamma \neq 0$ or 1

When the fractional parameter takes on the magnitudes within the interval $0 < \gamma < 1$, then the duration of contact $t_{\rm cont}^{(\gamma)}$ could be determined as follows

$$t_{\rm cont}^{(\gamma)} \approx t_{\rm cont}^{(0)} + \epsilon, \tag{27}$$

where ϵ is a small value.

Substituting (27) in equation

$$0 = \alpha = V_0 t - \frac{4}{15}\,æ V_0^{1/2} t^{5/2} + æ V_0^{1/2}\Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3} - \frac{1}{5}\,\gamma\right)\frac{t^{5/2+\gamma}}{(3/2+\gamma)(5/2+\gamma)} \tag{28}$$

yields

$$\epsilon = \frac{5}{2}\,\Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3} - \frac{1}{5}\,\gamma\right)\frac{t_{\rm cont}^{(0)\,1+\gamma}}{(3/2+\gamma)(5/2+\gamma)}.$$

Supposing that

$$t_{\max}^{(\gamma)} \approx t_{\max}^{(0)} + \epsilon_1, \tag{29}$$

where ϵ_1 is a small value, and substituting (29) in equation

$$0 = \dot{\alpha} = V_0 - \frac{2}{3}\,æ V_0^{1/2} t^{3/2} + æ V_0^{1/2}\Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3} - \frac{1}{5}\,\gamma\right)\frac{t^{3/2+\gamma}}{3/2+\gamma}, \tag{30}$$

we obtain

$$\epsilon_1 = \Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3} - \frac{1}{5}\,\gamma\right)\frac{t_{\max}^{(0)\,1+\gamma}}{(3/2+\gamma)}.$$

Now substituting (29) in (21) we could define

$$\alpha_{\max}^{(\gamma)} = \alpha_{\max}^{(0)} + \frac{9}{2}\,V_0\Delta_\gamma \frac{1}{\gamma}\left(\frac{1}{3} - \frac{1}{5}\,\gamma\right) \times \frac{t_{\max}^{(0)\,1+\gamma}}{(3/2+\gamma)(5/2+\gamma)}. \tag{31}$$

3.3 The case $\gamma = 1$

In the particular case $\gamma = 1$, the characteristic values take the form

$$t_{\rm cont}^{(1)} = t_{\rm cont}^{(0)} + \frac{4}{35}\Delta_1 t_{\rm cont}^{(0)\ 2}, \tag{32}$$

$$t_{\max}^{(1)} = t_{\max}^{(0)} + \frac{4}{25}\Delta_1 t_{\max}^{(0)\ 2}, \tag{33}$$

$$\alpha_{\max}^{(1)} = \alpha_{\max}^{(0)} + \frac{12}{175}\Delta_1 t_{\max}^{(0)\ 2}, \tag{34}$$

where $\Delta_1 = \Delta_\gamma|_{\gamma=1}$.

4 Conclusion

In the present paper, the problem on the normal impact of a viscoelastic spherical shell upon a rigid plate has been studied, when the damping features of the impactor are modelled by the fractional derivative standard linear solid model. An approximate analytical solution has been found.

The analysis carried out on the base of the suggested model allows us to make the following conclusion: maximal viscosity increases all values characterizing the process of shells interaction, $t_{\rm cont}$, $t_{\max}$, and $\alpha_{\max}$, since with the increase in the fractional parameter from zero to unit the viscosity enhances, resulting in the increment of the characteristic values from $t_{\rm cont}^{(0)}$, $t_{\max}^{(0)}$, and $\alpha_{\max}^{(0)}$ to $t_{\rm cont}^{(1)}$, $t_{\max}^{(1)}$, and $\alpha_{\max}^{(1)}$, respectively.

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