About Fourier Transform

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Abstract: - This paper analyses Fourier transform used for spectral analysis of periodical signals and emphasizes some of its properties. It is demonstrated that the spectrum is strongly depended of signal duration that is very important for very short signals which have a very rich spectrum, even for totally harmonic signals. Surprisingly is taken the conclusion that spectral function of harmonic signals with infinite duration is identically with Dirac function and more of this no matter of duration, it respects Heisenberg fourth uncertainty equation. In comparison with Fourier series, the spectrum which is emphasized by Fourier transform doesn't have maximum amplitudes for signals frequencies but only if the signal lasting a lot of time, in the other situations these maximum values are strongly de-phased while the signal time decreasing. That is why one can consider that Fourier series is useful especially for interpolation of non-harmonic periodical functions using harmonic functions and less for spectral analysis.

Key-Words: - signals, Fourier transform, continuous spectrum properties, Quantum Physics, Fourier series, discrete spectrum

1 Introduction

A signal is considered and interpreted as a variable that defines a time dependant physical phenomenon. As a generalized form, the signal is analytically defined, as a real function $f(t)$, with a single real variable that is t (=time). For this paper there were only selected several categories.

The best known and also the most commonly used periodical signals will be shortly presented below:

The signal $f(t) = A(t)\sin(\omega_0 t + \varphi)$ is a harmonic signal, where $A(t)$ is the amplitude, ω_0 is own pulsation, φ is initial phase; the signal is stationary for $A(t) = const$ and non-stationary for $A(t) \neq const$. If $A(t) = Ce^{-at}$, then the signal is called dampened signal, where a is the dampening factor[4].

The stationary non-harmonic signal can be found under many shapes, the most common of them being the "saw tooth" signal, originating from $f(t) = a(t - kT_0)$, the rectangular signal, originating from $f(t) = \begin{cases} a, & t \in [kT_0,\ kT_0 + \mu\ T_0) \\ 0, & t \in [kT_0 + \mu T_0,\ kT_0 + T_0) \end{cases}$, the "impulse train" signal that originates from $f(t) = \begin{cases} a, & t = kT_0 \\ 0, & t \neq kT_0 \end{cases}$ where T_0 is the own period $T_0 = 2\pi/\omega_0$, $k = \mathrm{int}(t/T_0)$, *int* is the "whole part of..." function, μ is a subunit and positive coefficient which characterizes the so-called filling factor of the signal, for μ=0,5 the signal has level *a* during the first half of the period and level *0* for the other second half.

If the amplitude (the maximum value of $f(t)$ signal on a *T* period) is constant in time, it means that the signals are stationary, if the amplitude is variable in time, the signals are known as non-stationary.

Time *t* is only considered for positive values, $t \geq 0$.

When talking about deterministic and non-periodical signals (signals that follow a known and reproducible rule and contain no repetitive sequences within their entire duration), we refer to three categories of signals that are by now classics:

- step signal $f(t) = \begin{cases} a, & t \geq 0 \\ 0, & t < 0 \end{cases}$,
- ramp signal $f(t) = \begin{cases} at, & t \geq 0 \\ 0, & t < 0 \end{cases}$ and
- impulse signal (or Dirac function), which could also be written as $\delta(t)$ and be defined in many ways for different technical needs.

We will use the simplest definition of the Dirac function, characterized by two simultaneous valid conditions $\delta(t)=\begin{cases}+\infty & t=a\\ 0, & t\neq a\end{cases}$, $\int_{-\infty}^{+\infty}\delta(t)dt=1$. Besides the above signals there are infinitely more non periodical signals that can be artificially produced or that may be received from the environment, such as the sound signal from a concert, where the signal rule is provided by the musical score and each instrument's specific resonance.

Random signals are those signals that cannot be described by a rule. That is why random signals cannot be considered determinist signals because there is no connection between cause (the rule) and effect (the signal) through within reproducible conditions. Such examples can be the evolution of the car's engine RPM during its running life, daily air temperature variation at a certain location and so on. Theoretically, some artificially produced functions can be interpolated with such random signals, given specific time limits $t\in[t_1,t_2]$ and the acceptance of a certain amount of inaccuracy, thus obtaining a deterministic signal. While this is the general approach for interpreting real life signals by deterministic signals, there are two issues that must be solved during this process:

-find the best adequate interpolation functions

-eliminate the errors and interference during the signal's acquisition.

All the signals described above are considered to be continuous signals, although mathematically speaking some of them, such as the saw tooth, rectangular or impulse signals offer obvious discontinuities. In spite of this, mathematical operations of integration, derivation and operational computation are done by ignoring the inherent inaccuracy. For some applications, such as Fourier series coefficients determination, only Dirichlet conditions are enough (the function which describes the signal is bound, has a finite number of discontinuities and finite extremes during it's period), conditions which are not fulfilled only by the impulse signal.

Mathematics creates the possibility that every stationary periodical non-harmonic signal with ω_0 pulsation to be interpolated with an infinite series of stationary harmonic functions whose pulsation is multiple of the signal pulsation $\omega=n\omega_0,\quad n=1,2,3\cdots\infty$. Based on this mathematical artifice, we are talking about Fourier series, one can explain the fact that any periodical non-harmonic signal has within sources able to stimulate a wide range of physical systems with own frequency equals any signal components $n\omega_0,\quad n=1,2,3\cdots\infty$ although not necessarily with equal signal pulsations ω_0. Also from Fourier series theory results that a pure harmonic signal doesn't contain signals with superior pulsations $n\omega_0$.

Due to this Fourier series exclusive interpretation applied to periodical signals, which shows that a stationary harmonic signal with a single pulsation ω_0 cannot contain components of other pulsations but its own pulsation ω_0, there is the opinion that stationary harmonic signals cannot stimulate physical systems which have own pulsations different from ω_0. The authors' research demonstrated that this theory is not true[3,5]. More than this, signals considered identical, that is with identical amplitudes, phases and pulsations, are not identical from spectral point of view if the signals have different durations and this can be noticed in many practical situations.

2 Fourier Series and Their Properties

A stationary periodical signal $g(t)$ with period $T_0=\frac{2\cdot\pi}{\omega_0}$, where ω_0 is the signal pulsation, which fulfils the Dirichlet conditions, can be represented by a mathematical series whose terms are harmonic functions with pulsations multiple of the ω_0 pulsation. The ω_0 pulsation is called fundamental pulsation and the harmonic function with the pulsation equal to ω_0 is called fundamental harmonic. Harmonic functions with pulsations $n\cdot\omega_0$, $n=2,3,\cdots$ are called *n* order harmonics. The general form for series of harmonic functions is:

$$f(t)=\sum_{n=0}^{\infty}\left[a_n\cdot\cos(n\cdot\omega_0\cdot t)+b_n\cdot\sin(n\cdot\omega_0\cdot t)\right] \quad (1)$$

where the series has an infinite number of members.

To actually identify the series means to know the a_n, b_n coefficients. In order to do that one needs put the condition that the series of harmonic functions (1) should have, within the continue domain of period *T*, the smallest square average deviations from the $g(t)$ function, respectively to satisfy the expression (2) for S = min, where :

$$S=\sqrt{\frac{\int_0^{T_0}\left\{g(t)-\sum_{n=0}^{\infty}\left[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right]\right\}^2 dt}{T}} \quad (2)$$

and so (3) has the minimum value:

$$\int_0^{T_0}\left\{g(t)-\sum_{n=0}^{\infty}\left[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right]\right\}^2 dt \quad (3)$$

Any integration limits can be chosen, under the condition to determine a time domain that includes an entire period of the $g(t)$ signal. The limits may be symmetrical, such as $\int_{-T_0/2}^{T_0/2}$, limits with asymmetrical form such as (2) and (3) or any other type of limits. One may use, especially in (3), a calculation domain as multiple of period T_0, but this doesn't necessarily bring more precision in the a_n, b_n coefficients determination.

To find out the a_n and b_n coefficients from (1) one needs to write the condition (3) under it's known form:

$$\begin{cases}\dfrac{\partial s}{\partial a_n}=0\\[2mm] \dfrac{\partial s}{\partial b_n}=0\end{cases} \quad (4)$$

By solving the equations system (4), one gets the expressions:

$$a_0=\frac{1}{T}\int_0^{T_0} g(t)\cdot dt; \qquad b_0=0; \quad \text{...........(5)}$$

$$a_n=\frac{2}{T}\int_0^{T_0} g(t)\cos(n\omega_0 t)dt; b_n=\frac{2}{T}\int_0^{T_0} g(t)\sin(n\omega_0 t)\cdot dt$$

where n= 1,2,...

Also in (5), there is no restriction in choosing the integration limits as long as one includes a complete period. Thus expression (1) has the form:

$$f(t)=a_0+\sum_{n=1}^{\infty}\left[a_n\cdot\cos(n\cdot\omega_0\cdot t)+b_n\cdot\sin(n\cdot\omega_0\cdot t)\right] \quad (6)$$

so that the constant a_0, which is a measure of signal $g(t)$ asymmetry in respect with the abscissa, is naturally separated from functions which represent harmonics of *n* order.

Decomposing a stationary signal $g(t)=A\cdot\sin(\omega_0\cdot t)$ in Fourier series, respectively calculating a_n, b_n coefficients, results in values $a_0=0$, $a_n=0, n=1,2\cdots$; $b_1=A; b_n=0, n=2,3,\cdots$ in other words, it results in the initial signal. That is where the opinion that a stationary harmonic signal has a single harmonic component, respectively itself, is coming from. This opinion will be invalidated in the following paragraphs.

The harmonic components were written as in (6), where each component is written like a two trigonometric functions sum, for reasons regarding a_n, b_n coefficients calculation. Grouping the two components of *n* order harmonic in this way, one obtains:

$$f(t)=a_0+\sum_{1}^{\infty}A_n\cdot\sin(n\cdot\omega_0\cdot t+\varphi_n) \quad (7)$$

where the amplitude A_n and phase φ_n result from the coefficients identification:

$$\begin{aligned}A_n\cdot\sin(n\cdot\omega_0\cdot t+\varphi_n)&=A_n\cdot\cos(\varphi_n)\cdot\sin(n\cdot\omega_0\cdot t)+\\+A_n\sin(\varphi_n)\cos(n\omega_0 t)&=a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\end{aligned} \quad (8)$$

we can thus identify the coefficients as:

$$\begin{cases}a_n=A_n\cdot\sin(\varphi_n)\\ b_n=A_n\cdot\cos(\varphi_n)\end{cases} \quad (9)$$

and we then get:

$$\begin{cases}A_n=\sqrt{a_n^2+b_n^2}\\[2mm] \varphi_n=arctg\dfrac{a_n}{b_n}\end{cases} \quad (10)$$

By using the (7) formula, with coefficients calculated according to relation (10), one finds that a specific order *n* harmonic is a stationary sinusoidal signal with it's own amplitude and phase. In expressions (7) and (10) one notices that a certain harmonic of order *n* is actually a signal produced by a rotating vector $\vec{v}_n$ (complex number), having modulus A_n and phase φ_n.

$$\vec{v}_n=A_n\cdot e^{j\cdot(n\omega_0 t+\varphi_n)} \quad (11)$$

Where components b_n are on the real axis and a_n are on the imaginary axis.

To determinate the harmonics of a periodic signal is not only a theoretical problem which allows decomposing a periodic function into other periodic functions. Harmonics existence are strongly felt in practice because a non harmonic periodic signal generates an infinite number of excitation sources having frequencies equal to multiples of the basic signal frequency and these sources produce obvious effects by stimulating physical systems with pulsations (frequencies) equal to any multiple of the signal's own pulsation (frequency)[5].

Frequency multipliers used in radio-technics are based solely on this obviously very real phenomenon. For multiplication, a stationary harmonic signal is distorted to generate harmonic components with pulsations $n\omega_0$ from which is extracted, through filtration, the component with the value *n* that is desired, usually $n = 2 \cdots 5$.

This phenomenon appears also is also found when referring to mechanical systems, respectively a non-harmonic signal generates sources of excitation which get in resonance with components of the system. A good example is given by non-harmonic signals such as earthquakes which produce damages to constructions or parts of constructions that have their own pulsation equal to harmonics of the earthquake.

From (7) one determines that a non-harmonic periodic signal is the better defined in Fourier series, the more components of the series are identified, respectively the higher *n* gets. In reality there is a limitation here: calculating the coefficients of the Fourier series, and even in the series itself, there are used the harmonic functions $\sin(n\omega_0 t), \cos(n\omega_0 t)$ and it is well known that functions $\sin(\infty), \cos(\infty)$ are indeterminate.

By using a computer to calculate the series elements the non determination situation described above is rapidly reached due to the way in which numbers are represented by the operating system or programming language, since numbers are only represented with a finite number of figures. The below examples clearly show this fact.

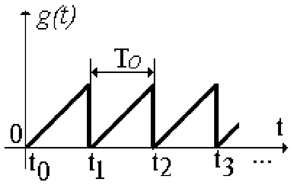


Fig. 1 - Tooth of the saw signal

In fig. 1- a saw tooth signal which is analytically defined in the above, having period $T_0 = 1$ second and *a=1,* fig. 1. In fig. 2, in the upper side we show the shape of the first eight harmonics and in lower side the initial signal, reconstructed from ten harmonics.

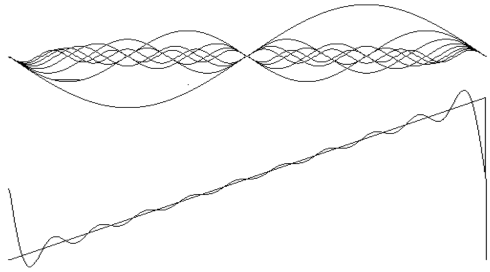
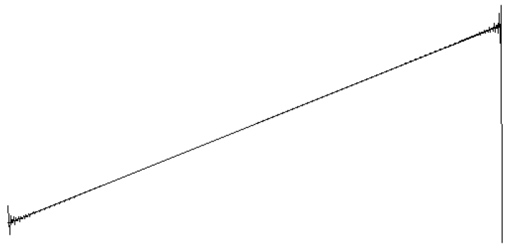


Fig. 2 - Reconstruction of ten harmonics

The value of the constant component a_o is 0,99401 and the amplitude of the tenth order harmonic A_{10} is 0,01593.

By increasing the number of harmonics one would expect the signal reconstruction to be more accurate, maybe with the exception of interval limits, where the mathematical discontinuity is also more pronounced. By increasing the number of harmonics, like in fig. 3 and 4, one notices a contradictory situation: reconstruction out of 300 harmonics is more precise than of 500, where large errors appear.


a_0=0,99401; A_{300}=0,00105
Fig.3 Reconstruction out of 300 armonics

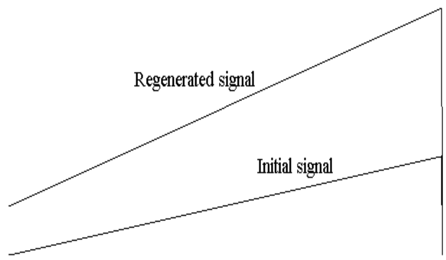


Fig. 4 - Reconstruction out of 500 armonics

To find the source of this deviation one has to research the evolution of the harmonic's amplitude in the studied cases, fig. 5.

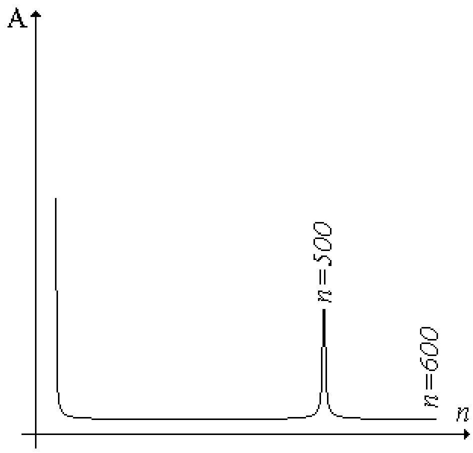


Fig. 5 - The amplitude of first 600 harmonics

If for a number up to 300-400 harmonics their amplitude continually decreases in the same time with the increase of the given harmonic's number of order, by continuing to increase the number of harmonics, their amplitude begins to increase again up to the value $A_{500} = 0{,}497$.

Continuing to increase the number of harmonics, their amplitude decreases again followed once more by an increase for higher values of order's number. This variation of the amplitude by the increase of the number of order, is not due to the structure of known relations of calculation but actually to the computer having to operate with hard conditioned relations. The cause of the deviations from figure 4, where the signal is reconstructed out of 500 harmonics, can be thus found in the particular way in which the digital computers operate with numbers which have a finite number of figures and, due to the truncation errors that appear when a large number of computations is being done, one may end up experiencing big errors.

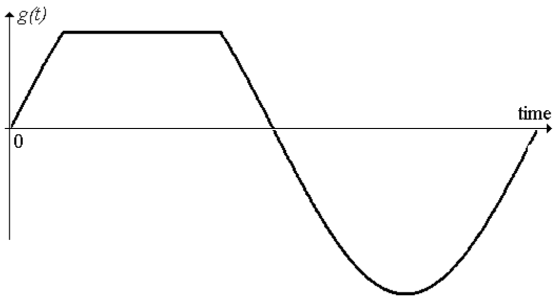


Fig. 6 – Distorted harmonic signal

Let's now analyze the signal from figure 6, which represents a harmonic signal of pulsation $\omega_0 = 10\, s^{-1}$, and described by the equation $g(t) = \sin(\omega_0 t)$, and is distorted through "cutting" in positive values domain between the angles of $\omega_0 t = 0.2\pi$ for $t = 0.2\,\pi/\omega_0$ and $\omega_0 t = 0.8\pi$ for $t = 0.8\,\pi/\omega_0$. The signal is only known during a period of time $\omega_0 t \in [0 \cdots 2\pi]$ for $t \in [0 \cdots 2\,\pi/\omega_0\,]$. The first 10 harmonics have the amplitudes described in table 2, where one can see that their values don't decrease continuously, but have a non-monotonous variation, their values being correctly evaluated because the evaluation of $\sin(n\omega_0 t), \cos(n\omega_0 t)$ was done for reasonably large values of the arguments.

For experimentally acquired signals, the upper limitation of the harmonics is resulting from considerations related to the signal sampling theory. Thus, should the signal be acquired at equally divided Δt time intervals, the maximum pulsation for which one can identify harmonic components of the Fourier series is given by:

$$\omega_{\max} = \frac{\pi}{\Delta t} \tag{12}$$

Table 1

The amplitude of the first 10 harmonics

n	1	2	3	4	5	6	7	8	9	10
ω	10	20	30	40	50	60	70	80	90	100
Ampl.	0,851	0,112	0,066	0,024	0,003	0,013	0,009	0,001	0,004	0,005

Subsequently one can say about Fourier series that:

- It is a series made of harmonic functions with pulsations equal with multiples of the pulsation of the non-harmonic periodic signal from which it originates;
- The harmonic functions which compose the Fourier series are real sources of excitation for physical systems which have own pulsations equal with one of the harmonic pulsation;
- Each harmonic's amplitude has a finite value, usually considered as continuously decreasing by the order of the harmonic, although there may occur situations where the decrease is not continued;
- The harmonic functions come together in a discrete spectrum of pulsations contained in the base signal;
- The characteristics of each spectrum's harmonic, meaning amplitude and phase, are independent from the signal duration;
- The Fourier series doesn't show if, among the discrete harmonic components, it has others able to excite various other oscillating systems.

3 About Fourier Transform

Non-harmonic periodical signals analytically described by a function which respects Dirichlet's conditions (the function which describes the signal is limited, has a finite number of discontinuities and finite extremes on the period's duration) are empirically known as numerical function and can be interpolated through an infinite series of harmonic functions, respectively Fourier series. If Fourier series is applied for periodical signals only, Fourier transform can be applied as well for periodical and non periodical signals which are considered as an extreme case of periodical signals. If the Fourier series identifies only discrete spectral components with pulsations equal with multiple of the pulsations of the periodical and non harmonic signal, the Fourier transform shows spectral components on the continuous domain of pulsations of a periodical or non periodical signal [1], [2], [3], [4]. Fourier transform makes that from a real function of time, which describes researched signal, to get a complex function having the pulsation as variable. This modulus of complex function is named either frequency characteristic if refers to attenuation properties of a medium whereby are transferred signals or spectral function if refers to spectral composition of a known signal [5]. The Fourier transform theory can be applied with the same results to any kind of signals [6] of very different technical field (electrical signals [7], radio electrical signals [8], [9], mechanical signals. Among Fourier transform results, the most surprising approaches to the Quantum Physics, mostly considered abstractly even for physicians.

Although Fourier transform theory is well known, this must to be shortly sum up for emphasize some useful rules.

A periodical signal $f(t)$ of period T_0, and pulsation $\omega_0 = 2\pi/T_0$, can be spectrally analyzed with the form:

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \quad (13)$$

called, by definition, Fourier transform of function $f(t)$ and is symbolically written:

$$F(j\omega) = F\left[f(t)\right] \quad (14)$$

The expressions of Fourier transforms were inferred for the case of non periodical signal. They are perfectly valid also for periodical signals, because can be ignored or considered as a no essential feature that the periodical signals values are regularly repeated at moments equal with one period.

Because of the form of expression (15) appears the necessity of knowing the real signal, depending of time, on an infinite domain, between plus and minus infinite. For a concrete signal, known between the finite moments t_1 and t_2, can be used the below argument:

$$f(t) = \begin{cases} 0, & t < t_1 \\ f(t), & t_1 \le t \le t_2 \end{cases} \Rightarrow$$

$$F(j\omega) = \int_{-\infty}^{t_1} 0dt + \int_{t_1}^{t_2} f(t) e^{-j\omega t} dt + \int_{t_2}^{+\infty} 0dt; 0, \quad t > t_2 \quad (15)$$

so :

$$F(j\omega) = \int_{t_1}^{t_2} f(t) e^{-j\omega t} dt$$

The transformations (13) and (15) make that a real function of real variable $f(t)$, to become a complex function $\left[F(j\omega)\right]$ with real and imaginary parts Re and Im. Using the writing ways of complex functions it gets:

$$\begin{aligned} Re\left[F(j\omega)\right] &= \int_{t_1}^{t_2} f(t) cos(-\omega t) dt; \\ Im\left[F(j\omega)\right] &= \int_{t_1}^{t_2} f(t) sin(-\omega t) dt \end{aligned} \quad (16)$$

The expression (16) allows finding the modulus (amplitude) S called hereinafter spectral function and phase ϕ of complex function as real functions of pulsation:

$$\begin{aligned} S(\omega) &= \sqrt{\left\{Re\left[F(j\omega)\right]\right\}^2 + \left\{Im\left[F(j\omega)\right]\right\}^2} \\ \phi(\omega) &= arctan \frac{Im\left[F(j\omega)\right]}{Re\left[F(j\omega)\right]} \end{aligned} \quad (17)$$

The definition domain of pulsation from the spectral function $S(\omega)$ is comprised between zero (negative pulsations don't make sense) and a maximum value ω_{max} which, for signals described by analytical functions, can be chosen of however high value depending on certain concrete criterion. For signals empirically taken as samples with constant period of time Δt, the value of ω_{max} is given by Shannon's sample theory, respectively:

$$\omega_{max} = \frac{\pi}{\Delta t} \quad (18)$$

We chose for the following analyze a pure harmonic signal which can exist alone or can appertain to a composite periodical signal:

$$f(t) = sin(\omega_o t) \quad (19)$$

which is quite known for $t \in \left[0, 2\pi/\omega_o\right]$. For the spectrum which is determined with Fourier transform using (15) and (17), the integration will be performed on variable durations between the limits $t_1 = 0$ and $t_2 = 2n\pi/\omega_o$ *(n* is a multiply of period $T_0 = 2\pi/\omega_0$) to have in view any influence of signal's duration, duration which doesn't appear at the Fourier series. After analytical calculating performing the expression for spectral function is getting:

$$S(\omega) = 2\omega_0 \left| \frac{\sin\left(n\pi \frac{\omega}{\omega_o}\right)}{\omega^2 - \omega_o^2} \right| \qquad (20)$$

4 Spectral Function Properties

The Fourier series gives for the signal (19) only a single spectral component, namely itself. The Fourier transform and the spectral function gives a large spectrum dependent of the signal duration n and presented in the figure 1 for $n=4$.

Analyzing the expression (20) is noticed some interesting issues:

- In point of abscise $\omega = \omega_0$, the amplitude has the value [11]:

$$S(\omega_0) = 2\omega_0 \lim_{\omega \to \omega_o} \left| \frac{\sin(n\pi\omega/\omega_o)}{\omega^2 - \omega_o^2} \right| = n\frac{\pi}{\omega_o} \qquad (21)$$

- The peak of frequency characteristic, respectively the maximum amplitude, appears for $\omega = \omega_0$ only if $n \to \infty$.
- The S amplitude in co-ordinate point $\omega = \omega_0$ is growing in the same time with signal's duration, so that for $n \to \infty$, $S \to \infty$.
- The signal (fig 7) contains an infinite number of pulsations existing within a $2d$ width loop around $\omega = \omega_0$ and a series of d width loops one side and another of the $\omega = \omega_0$.
- The loop centered on the value $\omega = \omega_0$ has the biggest amplitude, the other loops amplitude being smaller the farer they are of $\omega = \omega_0$; the loops width decreases as signal duration increases.

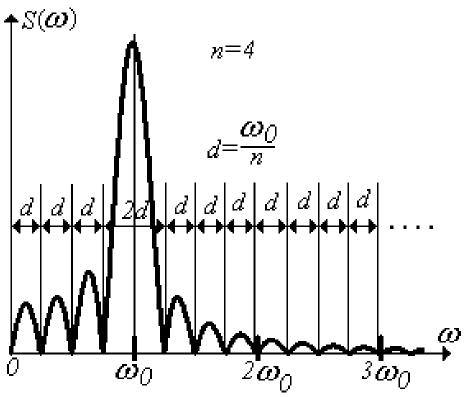


Fig. 7 Continuous spectrum of pure harmonic signal

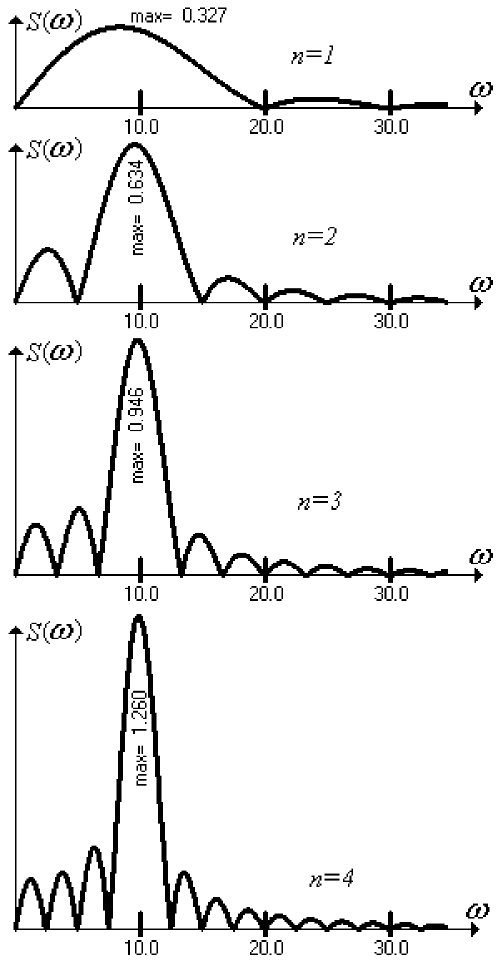


Fig. 8 Spectrum of signal with different duration

From those above a very interesting feature could be emphasized, figure 8 ($\omega_0 = 10\,s^{-1}$): as the pure harmonic signal is shorter, its spectrum is larger and the spectrum contains a very broad loop centered in $\omega = \omega_0$ called central loop and a series of loops with smaller and smaller amplitudes for pulsations farer and farer of $\omega = \omega_0$; as the signal duration increases the central loop width decreases and its amplitude increases and lateral loops became narrower and thicker, tending to get closer to the pulsation $\omega = \omega_0$.

If the signal is periodical but non-harmonic, than it contains a number of harmonic components which produces, each of them, a spectrum as is shown above, and each component's spectrum is cumulated into a global effect.

So it is explained why, in practice, an earthquake which takes for few ground oscillations causes walls, chimneys, pillars or many other construction elements collapsing with very different own pulsations.

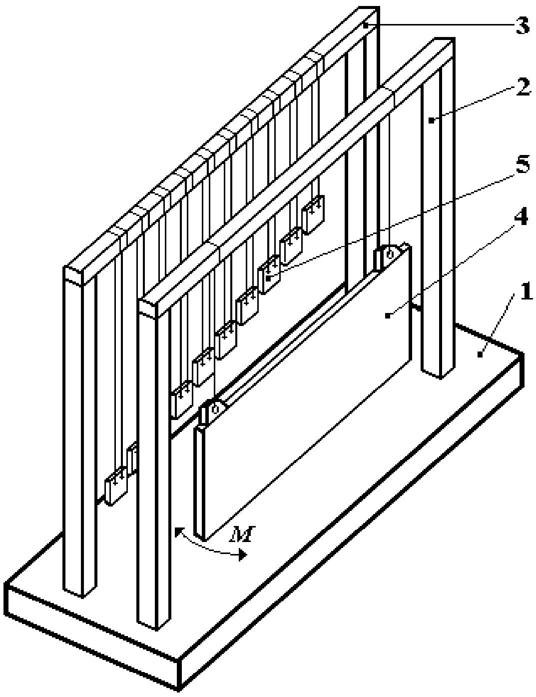


Fig. 9 Experimentally confirmation of the broad-band spectrum of the short signals

Even that a signal has a very long duration, its harmonics effect will have major variations while the signal proceeds over the object which is under its influence. In the beginning of respective signal's action, even after its first period will appear a lot of harmonics on a wide pulsations band, capable to stimulate a lot of oscillating systems with a wide variety of own frequencies and while the signal time of action is growing the harmonics tend to signal's own pulsation. In this way, the most dangerous time duration regarding the signal action is in its first periods, when the possibility to perturb is acting on the most wide spectrum possible.

Figure 9 shows a very simple device which emphasize all above. On a holder 1 are fixed two identically frames 2 and 3. On the frame 2 is suspended a plate 4 using two equal wires of length l and on the frame 3 are suspended a lot of plates 5 (we used 10) using wires of different lengths. l_i, $l_1 > l,\ \ l_5 = l,\ \ l_{10} < l$. So, the plates are forming some pendulums with different own pulsations, $\omega_i = \sqrt{g/l_i}$, g=9.81 ms^{-2}. For a ratio $l_1/l_{10} = 2$ the extreme pendulums will have pulsations in ratio $1 : \sqrt{2}$, resulting that $l_1/l_5 = l_5/l_{10} = \sqrt{2}$. The other pendulums length l_i will vary as a square root function to oscillate on pulsations with equidistant values between ω_1 and ω_{10}.

If plate 4 is moved under the direction M, it will oscillate with its own pulsation and will stimulate through pressure waves the pendulums 5. Will be noticed very easy that at first oscillation of plate 4 will be stimulated and will oscillate all pendulums 5 and then, when oscillations number of plate 4 is rising in time only pendulums 5 with length equals with plate 4 will continue to oscillate.

This mechanical device was chosen to emphasize the influence of signal duration upon the spectrum because it produces slow oscillations, easy to notice and to study. Can be made more other devices, also electronic devices but these are more complicated and hard to be analyzed.

5 Fourier Transform and Quantum Physics

In Quantum Physics exists Heisenberg's fourth equation of uncertainty [12], written like in following equation:

$$\delta W \cdot \delta\tau \geq h \tag{22}$$

Where δW is a wave energy variation, $\delta\tau$ is wave duration, h is Plank constant. If (22) is divided by h and taking into account that wave s energy is $W = h\nu$, ν =frequency, is getting:

$$\delta\nu \cdot \delta\tau \geq 1 \tag{23}$$

or

$$\delta\omega \cdot \delta\tau \geq 2\pi \tag{24}$$

This means that a signal spectrum has a pulsation variation range $\delta\omega$ inversely proportional to its duration $\delta\tau$:

$$\delta\omega \geq \frac{2\pi}{\delta t} \tag{25}$$

its spectrum being as broader and larger as it is shorter in time. This conclusion shows a first connection between Fourier transform and Quantum Physics.

To show another connection we have to weigh anchor the spectral function determination starting from the same pure harmonic signal (fig. 7) but integrated between negative and positive limits expressed as multiples of own period $T_0 = 2\pi/\omega_0$, to simplify the calculations:

$$F(j\omega) = \int_{-n_1T_0}^{n_2T_0} e^{-j\omega t} \sin(\omega_0 t)dt =$$
$$\int_{-n_1T_0}^{n_2T_0} \cos(\omega t)\sin(\omega_0 t)dt - j\int_{-n_1T_0}^{n_2T_0} \sin(\omega t)\sin(\omega_0 t$$
$$= \left[-\frac{\cos(\omega_0+\omega)t}{2(\omega_0+\omega)}\Bigg|_{-n_1T_0}^{n_2T_0} - \frac{\cos(\omega_0-\omega)t}{2(\omega_0-\omega)}\Bigg|_{-n_1T_0}^{n_2T_0} \right] - \quad (26)$$
$$-j\left[\frac{\sin(\omega_0-\omega)t}{2(\omega_0-\omega)}\Bigg|_{-n_1T_0}^{n_2T_0} - \frac{\sin(\omega_0+\omega)t}{2(\omega_0+\omega)}\Bigg|_{-n_1T_0}^{n_2T_0} \right] =$$
$$= Re\big(F(j\omega)\big) + j\,Im\big(F(j\omega)\big)$$

The calculations are simplified if $n_1 = n_2 = n$ for which $Re\big(F(j\omega)\big) = 0$, and spectral function becomes:

$$S(\omega) = 2\omega_0 \left| \frac{\sin(2\pi n\omega/\omega_0)}{\omega_0^2 - \omega^2} \right| \quad (27)$$

In co-ordinate $\omega = \omega_0$, (15) becomes:

$$S(\omega_0) = \frac{2\pi An}{\omega_0} \quad (28)$$

The spectral function is annulled in following co-ordinates:

$$\omega = \frac{k}{2n}\omega_0, \quad k = 0,1,2,3,\cdots, \quad k \neq 2n \quad (29)$$

A typical graph of spectral function for $n=2$ could be seen in figure 10 and it looks like graph form figure 1, the difference is that n represents the pairs number of periods the signal lasts.
The calculation of the each loop surface of this function, for an infinite signal duration, starts with the calculation of main loop surface, between ω_1 and ω_2, where $\omega_1 = \omega_0 - \omega_0/2n$, $\omega_2 = \omega_0 + \omega_0/2n$).

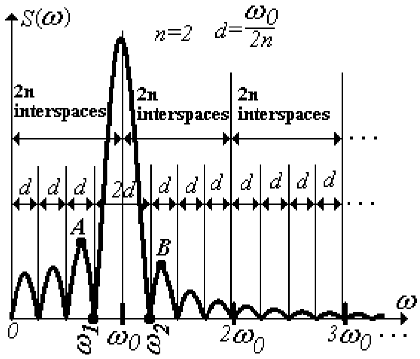


Fig. 10 Spectral function

Because $\int_{\omega_1}^{\omega_2} S(\omega)d\omega$ is transcendental can not be calculate on analytical way, so that it is solved using numerical method with data from table 2.

Table 2
Surface under the central loop as function of n

n	1	10	100	...	100000
Area	3.744....	3.7042....	3.7038....	...	3.70387410375....

It's noticed a very fast convergence to a transcendent number *3.70387410375...*, independent of ω_0.
The others loop surfaces, especially surfaces sum is very hard to be calculated with an analytical method and numerical calculations don't converge anymore, like in prior case. Majorising functions were used to solve this problem and also their convergence study for n infinite values. Because (27) is a modulus function, must find two majorising functions S_{M1} and S_{M2}:

$$S_{M1} = 2\omega_0 \frac{1}{\omega_0^2 - \omega^2}, \quad \omega < \omega_0 \quad (30)$$

$$S_{M2} = 2\omega_0 \frac{1}{\omega^2 - \omega_0^2}, \quad \omega > \omega_0 \quad (31)$$

The value $\omega = \omega_0$ is not contained in (30) and (31). The single loop surface is:

$$Area\ \ S_{M1} = 2\omega_0 \int_{\frac{k-1}{2n}\omega_0}^{\frac{k}{2n}\omega_0} \frac{d\omega}{\omega_0^2 - \omega^2} = = ln\frac{4n^2 + 2n + k - k^2}{4n^2 - 2n + k - k^2} \quad (32)$$

$$Area\ S_{M2} = 2\omega_0 \int_{\frac{k-1}{2n}\omega_0}^{\frac{k}{2n}\omega_0} \frac{d\omega}{\omega^2 - \omega_0^2} = ln\frac{k^2 + k + 2n - 4n^2}{k^2 + k - 2n - 4n^2} \quad (33)$$

From (29) results that for (32) $k < 2n - 1$ and for (33) $k > 2n$. Based on these, the arguments of logarithm function from (32) and (33) are positive and bigger than *1* for *k* and *n* finite values. At limit, these values became:

$$\begin{aligned}
&\lim_{k\to\infty} Area\ S_{M1} = 0 \qquad for\ any \quad n \\
&\lim_{n\to\infty} Area\ SA_{M1} = 0 \qquad for\ any \quad k \\
&\lim_{k\to\infty} Area\ S_{M2} = 0 \qquad for\ any \quad n \\
&\lim_{n\to\infty} Area\ S_{M2} = 0 \qquad for\ any \quad k
\end{aligned} \tag{34}$$

No one areas from (32) or (33) could be considered negative because the representative functions haven't negative values and taking into account the null values from (34) results that non-majorising functions values will be null, too. It can be drawn the conclusion that for a stationary harmonic signal of infinite duration ($n \to \infty$) and unitary amplitude, the total area of spectral function is independent of signal pulsation ω_0 and has a finite value, respectively *3.70387410375 ...* . For $n \to \infty$, the spectral function central loop in $\omega = \omega_0$ is narrowing to zero and tend to infinite amplitude. In this situation, the spectral function properties became:

$$\begin{cases}
\lim\limits_{n\to\infty} S(\omega) = \begin{cases} +\infty & for\ \omega = \omega_0 \\ 0 & for\ \omega \neq \omega_0 \end{cases} \\
\lim\limits_{n\to\infty} \int_0^\infty S(\omega)\, d\omega = 3.7038741037 \cdots
\end{cases} \tag{35}$$

In Quantum Physics, the Dirac function is symbolized with $\delta(t)$ and could be written like:

$$\begin{cases}
\delta(t) = \begin{cases} +\infty & for\ t = t_0 \\ 0 & for\ t \neq t_0 \end{cases} \\
\int_{-\infty}^{\infty} \delta(t)\, dt = 1
\end{cases} \tag{36}$$

where t_0 is a random value of time. It is noticed that the spectral function limit for an infinite duration signal has the same properties with Dirac function, with the exception of the constant *3.70387410375* .... instead of the unit.

6 Conclusions

Fourier transform represents an extension and a generalization of Fourier series, as is shown into the above demonstrations. If Fourier series can be applied only for periodical functions analysis under the condition that the period to be already known, the Fourier transform can be applied to any function, periodical or no periodical, when the period is not known. These two applications have in common the fact that both of them are providing information about spectral components of analyzed functions [13], [14], [15], [16]. About information's content, there are fundamental differences between these two applications. The most obvious and surprising difference appear in the most simple case of pure harmonic signal. If the signal is analyzed using Fourier series its spectrum contains only a single component or the signal itself. Analyzing the signal using Fourier transform one can get a spectrum with bands (figure 7) whose width depends of signal duration as number *n* of periods. So, the spectrum which Fourier series can get is a discreet one, while the Fourier transform provides a continue spectrum on many large or narrow bands depending of signal duration. Is very important to be noticed that for short signals, during only few periods, the continue spectrum has not the maximum value for its own pulsation ω_0 and that is why is very difficult to find the value of this spectrum frequency (figure 7). A very important case of short signals is represented by earthquake signals or some signals produced by industrial activities. For a correct evaluation, when one processes the measurements of spectrum determination or spectrum recording has to take into account the above aspects for the precise calculation of spectral components pulsations.

The facts are more complicated for the non-harmonic signals. A non-harmonic signal is formed very often by the sum of many harmonic signals with $\omega_{01}, \omega_{02}, \cdots \omega_{0n}$ pulsations of random values. In this case, an analysis using Fourier series will not be able to identify each components pulsation mean while Fourier transform is able to identify each $\omega_{01}, \omega_{02}, \cdots \omega_{0n}$ harmonics pulsation from spectral functions maximum under the condition that for each of them to be applied a correction given by signal duration.

An inconvenient of spectral function which results from Fourier transform is that for a longer signal the maximum value of each components spectrum is higher. One can say that the maximum values which indicate spectral components will not give the correct amplitude values of these components, because these values depending a lot of signal duration as (20) and (21) show.

For a spectral analysis of a non-harmonic periodical signal using Fourier series, the first difficulty to be passed with other methods is to find out the fundamental pulsation ω_0 of the signal. The spectral components have the pulsations $2\omega_0, 3\omega_0, \cdots n\omega_0$ indifferently of real structure of analyzed signal. That is why the Fourier series is especially used for periodical functions interpolation, as a mathematical artifice, than for spectral components analysis of a signal, as it was used.

Only Fourier transform can provide information regarding the spectrum, which compulsory is a continue spectrum, in case of non periodical signals. Because to solve the expressions (13)-(17) for experimental signals is very laborious, it was settled a very fast way of Fourier transform, called Fast Fourier Transform (FFT). Recently, normal or fast Fourier transform are implemented on different software such as LabVIEW [17] or MATLAB [18]. But, from the authors experience results that is not indicated to use so called brand software because could appear no-permitted errors. For example, a signal processed in LabVIEW will not emphasize the influence of signal time length, so the result suggests that the signal could be artificial extended to indicate only maximum values of spectral components but not the enough wide bands which appear with short signals, too. The calculation modern technique reaches a work velocity which allows to program expressions (13)-(17) without spending much time, getting correct results at once [19], [20], [21], [22], [23], [24]. As we already shown, this aspect is very important in very short signal case, but dangerous too, as earthquakes or some industrial activities effects.

If the signal duration is long enough in comparison with its own period than the signal's effect is variable in time. During its first period the signal produces the widest possible spectrum and is able to excite oscillating systems with a lot of own pulsations on a very wide band, figure 2. As long as the signal continues to exist its spectrum is narrowed to the signal's own pulsation. For the signals with low pulsation (as mechanical ones) this effect is very intensive because its spectrum is superposed over many mechanical systems own pulsations, especially in constructions field.

Investigating the properties of spectral function in a deeper manner the author found that for a pure harmonic signal with an infinite duration, the difference between spectral function and Dirac's function appears as a constant value which in Dirac's function case has the conventional value 1. This similitude of the spectral function with an abstract and conventional function from Quantum Physics is fulfilled by the fact that spectral function respects the Heisenberg fourth equation of incertitude, valid into all Physics micro-universe. It is possible that continuing the investigations to be discovered more other surprises which could give different meanings to the actual signal investigation techniques.

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