The Phase Ambiguity Resolution by the Exhaustion Method in a Single-Base Interferometer

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Abstract: In the paper, the phase methods are considered for the measurement of the spatial orientation of an object by means of satellite navigation equipment. Methods for the resolution of phase ambiguity are analyzed. Effectiveness and applicability of the one-step methods are discussed in more detail. It is proved that for the realization of the exhaustion method the minimal group of navigation spacecrafts should include 5–6 observed ones. When measuring signals of 8 spacecrafts of base length 1 m, an unambiguous solution is achieved practically in all cases.

Key-Words: − Satellite navigation, spatial orientation, resolution of phase ambiguity

1 Introduction

Modern satellite radio-navigation systems (SRNS's) determine the actual coordinates of an object within an accuracy of 3-5 m and a velocity vector within an accuracy of 0.1 m/s. Besides, the time-frequency binding equipment on the basis of the GLONASS/GPS radio-navigation systems is widely used. This enables one to synchronize an onboard time scale with the UTC (Universal Time Coordinated) scale within an accuracy of 100 ns.

The further development of radio-navigation equipment is related to the application of the phase methods for the measurement of parameters of navigation signals. The application of the phase methods in geodetic equipment allows determining the relative coordinates of an object accurate within centimeters and millimeters.

Another application of the phase methods is in determining the spatial orientation of an object from GLONASS/GPS signals. Unlike magnetic sensors and inertial systems for measuring the spatial orientation of an object, satellite goniometric equipment determines the angles of course, pitch, and roll with respect to the true meridian and is free of drift of the measured parameters. Besides, satellite goniometric equipment has small initialization time and is cheaper in comparison with inertial sensors.

The angular position of an object in space can be determined from SRNS signals on the basis of the measurement of difference of the course of a signal of a navigation spacecraft (NS) between the antennas located at the endpoints of base-vectors. To determine spatial orientation, it is sufficient to have two non-collinear base-vectors, i.e., three antennas not located on a straight line.

The phase shift of a signal of an NS taken by two spaced antennas and cosine of the angle between a base-vector and the vector directed towards an NS are related by the expression (Fig. 1):

$$\cos\alpha = \frac{\lambda \cdot \varphi}{2\pi B} \qquad (1)$$

where λ is the length of a wave of an NS signal; φ is the phase shift; B is the base length; α is the angle between a base-vector and the vector directed towards an NS.
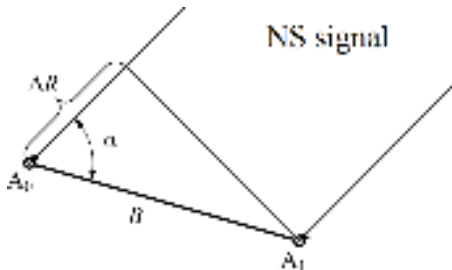

Fig. 1. One-base interferometer

The expression (1) is the equation of a one-base interferometer which is widely applied in the theory of phase direction-finders and antenna arrays.

The coordinates of a base-vector can be determined from the equation on the basis of the scalar product of vectors:

$$k_x x + k_y y + k_z z = \Delta R = \lambda \frac{\Phi}{2\pi} \qquad (2)$$

where k_x, k_y, k_z are the direction cosines of the vector directed towards an NS; x, y, z are base-vector coordinates; ΔR is path difference; Φ is phase shift of signals; λ is wave length.

To determine uniquely all unknowns, at least three equations are required. Considering that the coordinates of a base-vector are related to each other, with a base length B being known, the system of equations can be written in the form

$$\begin{cases} k_{x,j} x + k_{y,j} y + k_{z,j} z = \Phi_j \\ x^2 + y^2 + z^2 = B^2 \end{cases} \qquad (3)$$

where $j = 1, ..., N$ is the satellite number and the value Φ_j characterizes the individual right-hand side of (2) for j-th NS including λ_j and 2π.

The main problem in the phase measurements is in phase ambiguity. To improve accuracy of determining the spatial orientation, interferometers with a distance between antennas (base length) of several meters are used. Ambiguity of the measurement of phase shift is caused by the fact that the wave length of measured signals is rather small (about 19 cm) and is much less than the length of interferometer bases. The methods for the phase ambiguity resolution can be subdivided into two classes: the one-step methods that use the results of each measurement [1, 2] and the methods based on filtration which require the measurements of phase shifts during some time interval [3, 4]:

$$k_x x + k_y y + k_z z = \Phi_j + n_j \lambda_j \qquad (4)$$

where n_j is the integer phase ambiguity of the j-th NS.

Nowadays, in the majority of goniometric and geodetic equipment for the phase ambiguity resolution, the LAMBDA method [5, 6] is used. According to this method, at the first stage the integer ambiguity n is assumed to be an additional unknown without considering its integer value. Than each measurement gives one unknown n_j and for each separate measurement with N NS's the system of equations involves $N + 3$ unknowns. This results in lack of equations. Therefore the solution of a problem with the LAMBDA method requires, at least, two measurements taking into account that with measurements phase ambiguity does not vary. However, as the position of an NS varies slowly, with a motionless object, for each new measurement the equations (2) are strongly correlated with the previous ones and the system of equations, despite redundancy, is close to a degenerated one. To solve the obtained ill-condition system of equations, décor relation is applied. Thus, a part of the unknowns n_j can be estimated more or less reliably. At the second stage,

the obtained values of phase ambiguity are reduced to the integer form (generally by rounding-off), and further the initial system of equations is solved.

Note that currently the methods are actively developed which are based on simultaneous reception of signals from several navigation systems (GPS/Galileo/QZSS/SBAS [7, 8]) that allows one to improve the reliability of the phase ambiguity resolution. Also, the use of multi-antenna interferometric systems can significantly simplify the process of resolving an initial phase ambiguity [9, 10].

But these methods have some disadvantages: they are mainly used for short baselines (up to 1 m) and have a long convergence time.

2   The exhaustion method

One of the priorities in determining the spatial orientation of objects by SRNS is to improve significantly the measurement accuracy and to decrease the time of convergence of a solution simultaneously. The one-step methods for the phase ambiguity resolution [11–13] are of special interest. In these methods, on the basis of maximum likelihood, redundancy of the system of equations is used which can be obtained with the help of a redundant constellation of NS's. These methods admit resolution of ambiguities with high reliability during one stroke. Besides, solving the problem is possible not only for short bases but also for long ones up to 10 m.

In the one-step method for the phase ambiguity resolution in a one-base interferometer, the exhaustion method is used. The solution is taken by the maximum likelihood criterion. When taking in the signals of N NS's, the likelihood function (LF) is of the form

$$W\left(\Phi_1, \Phi_2, \ldots, \Phi_N \mid x, y, z\right) = \prod_{j=1}^{N}\left[\frac{1}{\sigma_j\sqrt{2\pi}}\right]$$

$$\cdot \exp\left[-\sum_{j=1}^{N}\frac{\left[\Phi_j + n_j\lambda_j - \left(k_{x,j}x + k_{y,j}y + k_{z,j}z\right)\right]^2}{2\sigma_j^2}\right] \quad (5)$$

with the additional condition

$$x^2 + y^2 + z^2 = B^2. \quad (6)$$

The function (5) can have local minima for any combination of ambiguities n_j. The problem of minimization of the likelihood function with respect to all possible values of n_j is solved by exhaustive search. The main disadvantage of this minimization method is in the large number of combinations of ambiguities n_j. When taking in the signals of N NS's, the number of combinations of ambiguities is $n_{max}^N = \text{int}\left(2B/\lambda + 1\right)$ where $\text{int}(s)$ means the integer part of a real s. For example, with a base of length $B = 1$ m an ambiguity n_j with respect to each NS can take 11 values (from –5 to 5). The total number of combinations of ambiguities is $11^3 = 1331$ for measurements for three NS's, $11^4 = 14641$ for four NS's, and $\approx 2 \cdot 10^8$ for eight ones. To each combination of n_j there corresponds a local minimum of the function (5). For a large number of combinations of n_j, the values of local minima could become close to a global minimum. Hence, a false decision may be taken. For the phase ambiguity resolution, to reduce the number of calculations, we can decrease the base length. However, this results in poor accuracy of angular measurements.

The number of calculations can be reduced considerably provided that an initial groupment with the minimal number of NS's (nonredundant groupment) is taken. Searching through all possible combinations of phase ambiguity and solving the problem for these values of phase ambiguity, we obtain an initial set of solutions. Further each solution of the initial set is verified by the solution for the complete groupment. In addition, false solutions are rejected by the maximum likelihood criterion, or, which is the same, by

the admissible total residual of a solution of the least squares minimum (LSM).

The potential of the exhaustion methods can be studied by the analysis of the likelihood function. The angular position of a base-vector for known length can be defined by two parameters: the course angle K and the pitch angle Ψ. Hence, the likelihood function is two-dimensional. The course and pitch angles are related with the rectangular coordinates by the expressions

$$X = B \cdot \cos K \cdot \cos \Psi ;$$
$$Y = B \cdot \sin K \cdot \cos \Psi ; \ \ Z = B \cdot \sin \Psi . \tag{7}$$

When resolving phase ambiguity, the probability of rough errors, i.e., the cases where phase ambiguity is determined with errors, is of special interest. Rough errors arise when the LF has subordinate maxima which are comparable in magnitude with the main maximum corresponding to a correct solution. Such a case is shown in Fig. 2 where the likelihood function for one NS is presented. We see that the phase ambiguity resolution is impossible with the measurements for one base for each NS separately since the likelihood function takes its extremum in whole areas and false solutions are indistinguishable from the correct one.
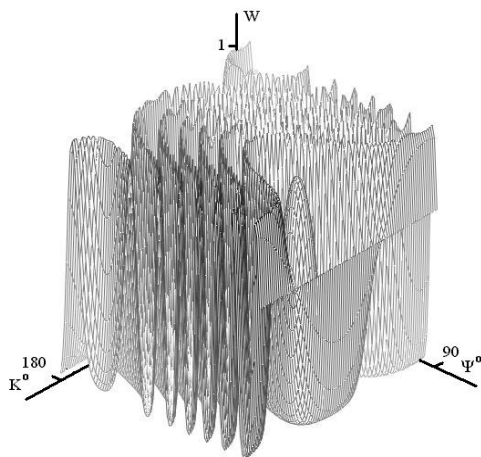

Fig. 2. The LF with the measurements for one NS

With increasing the number of observed NS's, the total residual is the sum of wave-like functions,

obtained with the measurements for each NS, and is the result of their interference. In Fig. 3, the LF for four observed NS's is presented. Here the main and subordinate maxima can be clearly distinguished.
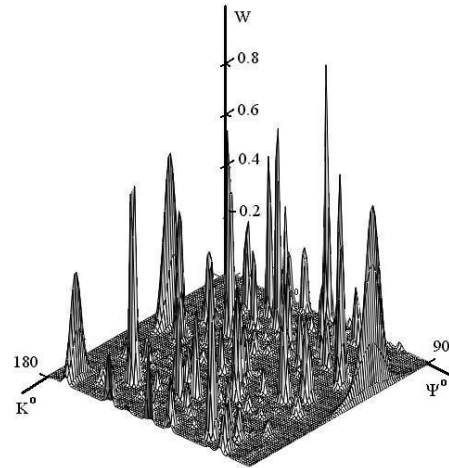

Fig. 3. The LF with the measurements for four NS's

The LF is rather complicated for analysis therefore we have to introduce a parameter that enables us to estimate probability of missing a correct solution and probability of rough errors, i.e., probability of taking a false solution. As such a parameter, we can take the index of the LF being the total residual of a solution of the LSM, which is equal to the sum of squares of residuals for all NS's or the square root of this value.

The residuals have two components: one of them is caused by residuals in subordinate maxima due to taking a false solution and another one is due to dispersion of the measured phase shifts. A subordinate maximum of the LF provides a false solution, thus, in the case of redundancy the system of equations becomes incompatible even for zero values of the error of the phase shift measurement. Without measurement noise, the magnitude of a residual in a subordinate maximum of the LF depends on the shape of an NS and on the values of phase ambiguity. Hence, this value can be considered to be mathematical expectation of residuals. The noise error of the phase shift measurements has the normal distribution. Thus, when solving the system (3), residuals

for each NS in the main and subordinate maxima of the LF are distributed by the normal law with mathematical expectation equal to residuals arising in the absence of measurement noise.

We consider the probability distribution function for the total residual. Provided that mathematical expectations of the quantities x_j are zero and their dispersions are equal to each other, the quantity $z = x_1^2 + x_2^2 + ... + x_N^2$ is distributed by the χ^2 law with n degrees of freedom [14]. This takes place in the main maximum of the LF for the phase shift measurements with the same accuracy. Mathematical expectation in a subordinate maximum is not equal to zero and the χ^2 distribution law in the classical form cannot be applied.

The distribution function for the total residual can be obtained as follows. First we obtain probability density for the square of one random variable. Then by the rule of addition of random variables we can obtain the required probability density. To calculate the distribution function for the square of the total residual we use the characteristic functions [15].

We can show that the characteristic function of a squared random variable with nonzero mathematical expectation is of the form

$$\Theta(v) = \frac{1}{\sqrt{1 - 2i\sigma^2 v}} \cdot \exp\left(\frac{im^2 v}{1 - 2i\sigma^2 v}\right). \qquad (8)$$

The characteristic function of the sum of squares of independent normal random variables with nonzero averages is equal to the product of the characteristic functions of the terms:

$$\Theta_n(v) = \left(1 - 2i\sigma^2 v\right)^{-\frac{n}{2}} \cdot \exp\left(\frac{iv}{1 - 2i\sigma^2 v} \sum_{k=1}^{n} m_k^2\right). \qquad (9)$$

The expression (9) implies a property of the characteristic function: it depends not on mathematical expectations of initial random variables but on the sum of their squares $m^2 = \sum_{k} m_k^2$. The distribution function for the sum of the squares of residuals must have the same property.

The probability density can be obtained by the inverse Fourier transform of the characteristic function (9):

$$P_n(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \Theta_n(v) \cdot e^{-ivx} dv$$

$$= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \left(1 - 2i\sigma^2 v\right)^{-n/2} \cdot \exp\left(\frac{ivm^2}{1 - 2i\sigma^2 v}\right) \cdot e^{-ivx} dv \qquad (10)$$

The graphs of probability density for different values of m for five observed NS's are presented in Fig. 4. From the graphs in Fig. 4 it follows that false solutions for $m > 5\sigma$ can be rejected.
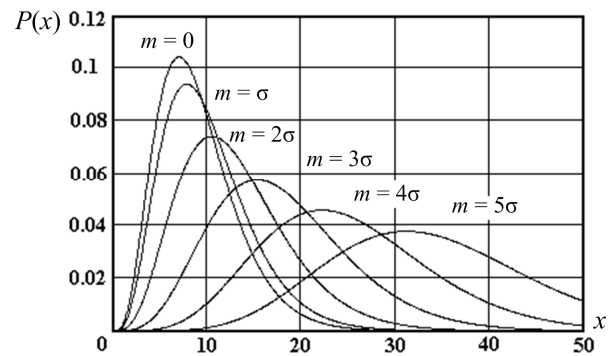

Fig. 4. Probability density for the total residual for different values of m

The probability density (10) may not be expressed in terms of elementary or tabulated functions. This complicates the calculation of probability of a given quantity being in a specific area. To this end, we can obtain the integral distribution function

$$W_n(x) = \frac{1}{2\pi} \int_{-\infty}^{x} \int_{-\infty}^{\infty} \Theta(v) \cdot e^{-ivx} dv dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Theta(v)}{iv} \cdot \left(1 - e^{-ivx}\right) dv \qquad (11)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left(1 - 2i\sigma^2 v\right)^{-n/2}}{iv} \cdot \exp\left(\frac{im^2 v}{1 - 2i\sigma^2 v}\right)\left(1 - e^{-ivx}\right) dv.$$

Given probability of rejecting a correct solution, to determine probability of a false solution it is necessary to determine the threshold value for which a correct solution falls into the list of possible solutions with given probability. The threshold value can be taken with the help of the integral distribution function (11) putting $m = 0$.

The dependence of probability of false solutions on the ratio of the total mathematical expectation of residuals to their mean-square deviation (MSD) for different number of observed NS's is shown in Fig. 5. The probability of taking a false solution is considerably defined by minimal mathematical expectation of the total residual in subordinate maxima. Fig. 5 shows that efficiency of rejecting a false solution is achieved for $m > (5 \cdots 6)\sigma$.
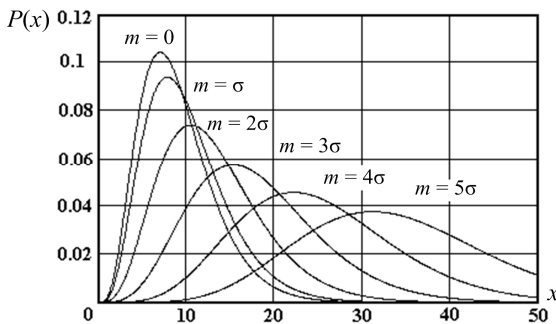

Fig. 5. The dependence of probability of a false solution on the ratio of the total mathematical expectation of residuals to their MSD

With exhaustive search of possible solutions, we obtain a set of components of residual due to the wrong phase ambiguity resolution. The components are deterministic variables and represent mathematical expectation m of residuals. They can be calculated a priori for each combination of phase ambiguities. The minimum value of these quantities is of special interest since in this case probability of a false solution is maximal and with increasing m

probability of a false solution decreases rapidly. However, the calculation of residuals for each specific case presents considerable difficulties, mainly because of a large number of combinations of phase ambiguity which take place in exhaustive search of all alternatives.

In the analysis, mathematical expectations of residuals in subordinate maxima (for zero errors of the phase shift measurements) can be considered as random variables. According to the obtained data, the distribution of mathematical expectations of residuals does not depend on the shape of the constellation of NS's, a space position, and the length of a base-vector. However, the minimal value of the total residual decreases with increasing the length of base. This dependence is explained by the quadratic increase of the number of possible positions of a base-vector with increasing its length. The square root of the sum of squares of residuals (the total residual) is reasonably described by the normal distribution and the mean-square deviation does not depend on the number of NS's in a groupment and equals to 28 mm. The considered number of NS's in a groupment varies from 4 to 13 for different positions and the length of a base-vector. The case with the measurements for 4 NS's is an exception. Histograms of distributions for $n = 4$ and $n = 9$ are shown in Fig. 6. The normal distribution for a large number of NS's can be explained by a consequence of the central limit theorem. With the number of NS's greater than 5, the average value of the total residual depends linearly on the number of NS's in a groupment.

The probability of taking a false solution is mainly defined by the minimal residual in subordinate maxima. Using the expression (11) for the integral distribution function, we can determine probability of taking a false solution. In Fig. 7, probabilities of a rough error depending on the error of the phase shift measurements are illustrated for the minimal total residual and a base of length 1 m and 10 m.
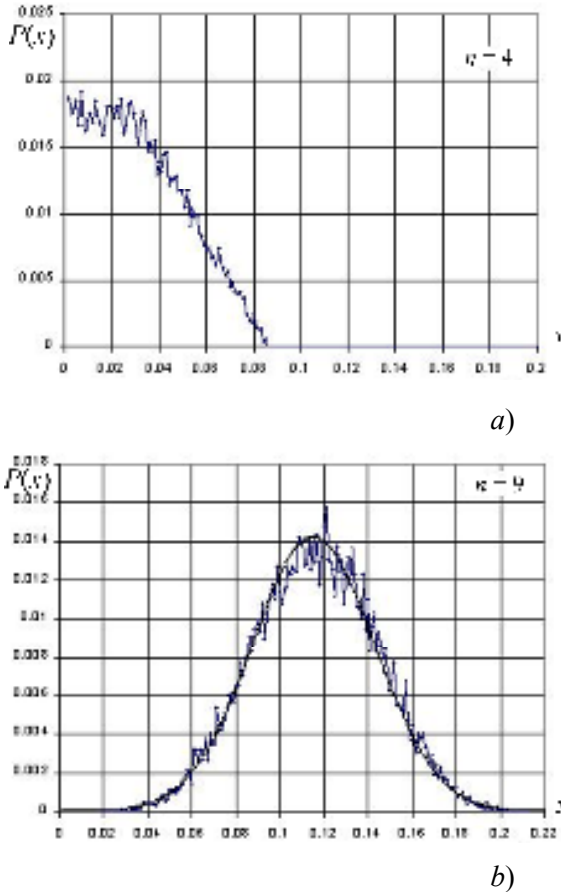
a)

b)

Fig. 6. Histograms of the distribution of the total residual
for four (*a*) and nine (*b*) observed NS's

3 Conclusion

From the results of the studies we can make the following conclusions.

1). Efficiency of the exhaustion method for the phase ambiguity resolution depends on the number of the observed NS's and the length of a base. For the base of length 1 m, the exhaustion method can work successfully for 5 observed NS's and for a noise error of the phase shift measurements equal to 5°. Nevertheless, for the base of length 10 m and for the same error of the phase shift measurements, 7-8 observed NS's are required.

2). The exhaustion method for the phase ambiguity resolution can be applied for an interferometer base of length up to 3 m and the limit MSD error of the phase shift measurements from 15 to 20°.

3). The minimal constellation of NS's for the implementation of the exhaustion method consists of 5–6 observed NS's. For the measurements of signals of 8 NS's and for a base of length 1 m, we obtain a unique solution in almost all cases.
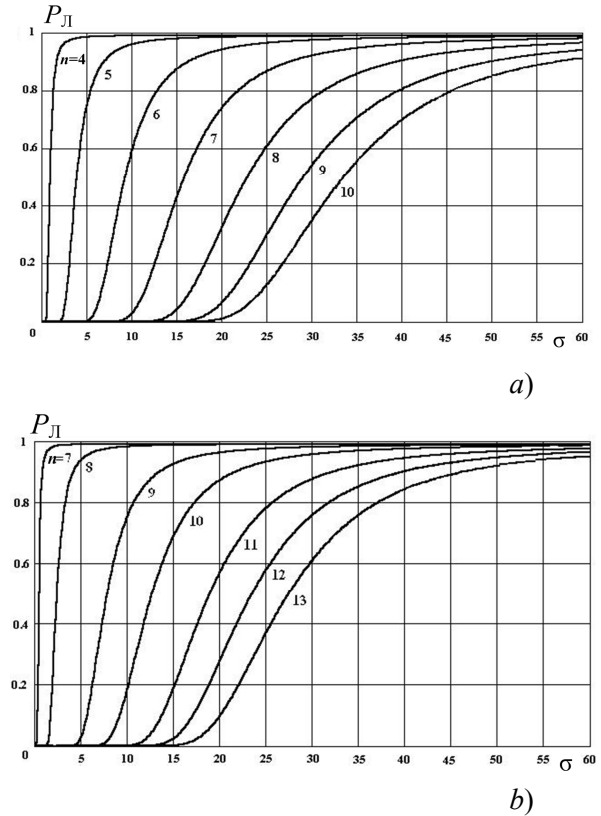

a)

b)

Fig. 7. Probability of a rough error
for a base of length 1 m (*a*) and 10 m (*b*)

Note that in practice the one-step exhaustion method for one base is applied to form an initial set of solutions. Hence, the probability of rejection of a correct solution (which is defined by the threshold value of the likelihood function) is an important property. The presence of false solutions in an initial set does not mean a rough mistake provided that a correct solution is involved as well. Further rejection of false solutions can be carried out by the filtration of solutions of an initial set or by the use of a multi-base antenna system.

Acknowledgement

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