The Overcome of a Priori Information Problem in Sampling-Reconstruction Procedures of Gaussian Process Realizations

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Abstract: - The case of the Sampling-Reconstruction Procedure of Gaussian process realizations is investigated when a priori information about some parameters of a given process is unknown. The general case with unknown expectation, variance and covariance function is considered. The unknown parameters are estimated on the base of the received set of samples. In result we have some adaptive algorithms. The method of the investigation is founded on the mathematical simulations. The results of the investigation demonstrate that the covariance function has the great influence on the main characteristics of the Sampling-Reconstruction Procedure. So the measurement of the covariance function is the most important operation in the adaptive algorithms.

Key-Words: - A Priori Information Problem, Gaussian Process, Sampling - Reconstruction Procedure.

1 Introduction

In signal processing, the problem of a statistical description of Sampling - Reconstruction Procedure (SRP) of random process is very important. The classic theorem was proposed mainly by Whittacker, Kotelnikov and Shannon, which is also known as WKS theorem. This theorem is valid only for deterministic processes.

In 1957, Balakrishnan [1] generalizes the WKS theorem for realizations of stationary random processes with restricted power spectrum. There are many publications devoted to these problems (see for example [2], [3]).

Balakrishnan´s theorem and its generalizations have some disadvantages in the description of Sampling – Reconstruction Procedure (SRP) of random process realizations: 1) reconstruction function is linear and the same for all types of random processes, 2) the probability density function is not used, 3) the number of samples must be infinite, 4) reconstruction error equal to zero, and 5) the both principal SRP characteristics (the reconstruction function and the reconstruction error function) do not depend on the probability density function, on a covariance function or on a type of power spectrum.

To overcome these disadvantages and to demonstrate the influence of the above mentioned reasons to the SRP, we need to apply a different methodology for the description of SRP problem. This methodology is based on the Conditional Expectation Rule (CER) (see for instance [4]). The first application of this rule for the statistical description of SRP has been published in [5]. This methodology has been used in many publications (see for example [6] - [14] and the literature cited within). SRP of realization of many types random processes have been investigated on the basis of CER. For each type of process optimal reconstruction algorithm and error reconstruction function were obtained. But we have to emphasize that all mathematical models of random processes are completely described, or in other words all statistical characteristics of these processes were known.

In practice, the statistical description of random processes may not be complete. In these cases one or some parameters of the given process are unknown. Such problem is known as a priory information problem. To solve such SRP problem it is necessary to measure these parameters using the set of samples. With any next arrived sample the measurement results must be entered into the formula instead of the actual parameter value. The measurement results are changed with each step, so the SRP must have *a non-stationary* character. Meantime the sampled realization can be taken from *the stationary* random process. This is a special property of the problem. Such algorithms are called as adaptive algorithms.

The main goal of the present paper is connected with the investigation of the transition regimes in the adaptive SRP algorithms of various types. We demonstrate that the measurement of the covariance function is the most important operation in the algorithms under consideration.

The rest of the paper is organized as follows. In Section II, we describe the main features of the deterministic algorithm when the Gaussian process is completely known. A brief description of the proposed method for unknown parameter estimation is given in Section III. Section VI describes the adaptive algorithm for a simple case when the mathematical expectation is unknown. The general adaptive algorithm is described in Section V when all parameters are unknown. The results are given in sections VI, VII and VIII, respectively, when the mathematical expectation, the variance and the covariance moment is unknown. Finally, Section IX concludes this paper.

2 Main Features of the Deterministic Algorithm

Let us suppose that the statistical description of the given Gaussian process is completely known. Generally, this process is not stationary. Then, all parameters (the mathematical expectation $m(t)$, the variance $\sigma^2(t)$ and the covariance function $K(t_1,t_2)$) are known. In addition, the set of samples $X,T=\{x(T_1),\ x(T_2),\ \ldots,\ x(T_N)\}$ must be known as well. The locations of samples and their number N are arbitrary. In this case the reconstruction function $\tilde{m}(t|X,T)=\tilde{m}(t)$ or the conditional expectation is expressed by the formula [15]:

$$\tilde{m}(t)=m(t)+\sum_{i}^{N}\sum_{j}^{N}K(t,T_i)\,a_{ij}\left[x(T_j)-m(T_j)\right] \quad (1)$$

where a_{ij} is an element of the inverse covariance matrix of the process.

The elements a_{ij} are calculated for the sample times T_i, T_j. The error reconstruction function or the conditional variance function is characterized by the formula [10]:

$$\sigma^2(t)=\sigma^2(t)-\sum_{i}^{N}\sum_{j}^{N}K(t,T_i)\,a_{ij}K(T_j,t)\,. \quad (2)$$

When the given process is stationary, the formulas (1) and (2) are simplified:

$$\tilde{m}(t)=m+\sum_{i}^{N}\sum_{j}^{N}K(t-T_i)\,a_{ij}\left[x(T_j)-m\right], \quad (3)$$

$$\sigma^2(t)=\sigma^2-\sum_{i}^{N}\sum_{j}^{N}K(t-T_i)\,a_{ij}K(T_j-t)\,. \quad (4)$$

Let assume:

$$m=0, \quad \sigma^2=1. \quad (5)$$

Then, instead of (3), (4) we have

$$\tilde{m}(t)=\sum_{i}^{N}\sum_{j}^{N}K(t-T_i)\,a_{ij}x(T_j)\,, \quad (6)$$

$$\sigma^2(t)=1-\sum_{i}^{N}\sum_{j}^{N}R(t-T_i)\,a_{ij}R(T_j-t)\,, \quad (7)$$

here $R(\cdot)=\sigma^{-2}K(\cdot)$ is the normalized covariance function.

From (1) – (7) one can see two remarkable points: 1) the reconstruction functions (1), (3) and (6) are linear functions with respect to received samples; 2) in all cases (2), (4) and (7) the conditional variance does not depend on sample values, but depends on the location of these samples.

3 Expressions for Unknown Parameter Estimation

Below we consider the variant when the covariance function is unknown. In this case we need to estimate this parameter. We use a simple method to obtain the expected result. We introduce "k" to designate the number of samples, which are involved in the operation of the parameter estimation. It is necessary to emphasize that the values N and k are generally different. Equality $k=N$ can be, in the particular case, when all samples involved in the realization reconstruction. This case is theoretical. Usually $N < k$, for example, in the Markov case $N=2$, but the value k may have a big number. We write the desired well known formulas.

The estimate of the expected value:

$$\hat{m}^{k}=\frac{1}{k}\sum_{l=1}^{k}x(T_l)\,. \quad (8)$$

The estimation of the variance:

$$\left[\sigma^2\right]^{k}=\frac{1}{k-1}\sum_{l=1}^{k}\left[x(T_l)-m\right]^2 \quad (9)$$

The estimate of the normalized covariance moment $R^{k}(p\Delta T)$ with a discrete argument $p\Delta T$ ($p=1,2,\ldots$) is:

$$R^{k}(p\Delta T)=\frac{1}{\sigma^2}\frac{1}{k}\sum_{l=1}^{k}\left[x(T_l)-m\right]\left[x(T_{l+p})-m\right]\cdot \quad (10)$$

where ΔT is a constant given by $\Delta T=T_i-T_{i-1}$.

4 Adaptive Algorithm for a Simple Case

Let us suppose that the unknown parameter is the mathematical expectation m of the stationary process. We estimate it by the formula (8). At each step k we have a different value of the mathematical expectation m^k. It is necessary to insert m^k in the formula (3) instead of the value m for the conditional expectation. Then the new function that describe the reconstruction depends on m^k, i.e.:

$$m^k(t) = m^k + \sum_i^N \sum_j^N K(t - T_i) a_{ij} \times \times \left[x(T_j) - m^k(T_j)\right]. \quad (11)$$

It is clear that

$$m(t) \neq m^k(t). \quad (12)$$

This fact is the source of another part of the reconstruction error. Let's call this as "an additional error" and emphasize that this error is a conditional function with respect to the set of samples X, T. Taking into account this fact, we write the *instant* mistake $\tilde{\delta}^k(t)$:

$$\tilde{\delta}^k(t) = \tilde{x}(t) - m^k(t) \quad (13)$$

here $\tilde{x}(t) = \left[x(t) \mid X, T\right]$ is the realization of the conditional process.

Let us square both parts of (13) and calculate the statistical average operation using a conditional probability density function. In result we have the general formula of the reconstruction error:

$$\left\langle \left[\tilde{\delta}^k(t)\right]^2 \right\rangle = \left\langle \left[\tilde{x}(t) - m^k(t)\right]^2 \right\rangle. \quad (14)$$

The first term of the right part of (14) is:

$$\left\langle \tilde{x}^2 \right\rangle = \sigma^2(t) + \left[\tilde{m}(t)\right]^2. \quad (15)$$

Taking into account (15) one can write (14) in the form:

$$\left\langle \left[\tilde{\delta}^k(t)\right]^2 \right\rangle = \sigma^2(t) + \left[m(t) - m^k(t)\right]^2. \quad (16)$$

The expression (16) involves two parts: the first is the usual conditional variance [see (2), (4), (7)], the second part is the additional reconstruction error occurred in the result of the inequality (12).

The main goal of the present work is connected with the investigation of the additional part in (16). We designate this as:

$$\left[\varepsilon^k(t)\right]^2 = \left[\tilde{m}(t) - m^k(t)\right]^2. \quad (17)$$

It is clear, when $k \to \infty$ the value $m^k(t) \to \tilde{m}(t)$. Finally, the additional error tends to zero and the reconstruction error will be determined by the conditional variance only.

One can remind that the conditional variance (or the usual error reconstruction function) does not depend on the values X,T of samples, but the additional error function (17) depends on values of samples. So, we need to calculate this part of the total error with the base of the set of samples X, T, which must be taken from a *simulated* realization.

5 The General Adaptive Algorithm

The formulas (11) - (17) can be modified for the general case, when all three parameters of a given Gaussian process are unknown: the mathematical expectation, variance and normalized covariance function. Again, on the basis of a sample set and formulas (8) - (10) we need to find the estimations of these parameters. Then introducing these estimations in the correspondent formulas, one can obtain the reconstruction function and the reconstruction error function in the transition regime.

Now instead of (11) we have:

$$m^k\left(t; \hat{m}^k, \left[\sigma^2\right]^k, \hat{R}^k\right) = m^k + \sum_i^N \sum_j^N \left[\sigma^2\right]^k \times \hat{R}^k(t - T_i)\, \hat{a}_{ij}^k \left[x(T_j) - m^k(T_j)\right]. \quad (18)$$

In (18) the elements of the inverse covariance matrix $\hat{a}_{ij}^k$ depend on the number of step k by two factors: the estimation of the variance and the estimation of the normalized covariance moment. It is clear, because each estimated element of the covariance matrix is determined by the following formula:

$$\hat{K}^k(T_i - T_j) = \left[\sigma^2\right]^k \hat{R}^k(T_i - T_j). \quad (19)$$

Once again, because the reconstruction function (18) does not coincide with the ideal reconstruction function, an additional error is occurred. By analogy with (17) we write:

$$\left[\varepsilon^k\left(t; \hat{m}^k, \left[\sigma^2\right]^k, \hat{R}^k\right)\right]^2 = \left[\tilde{m}(t) - m^k\left(t; \hat{m}^k, \left[\sigma^2\right]^k, \hat{R}^k\right)\right]^2. \quad (20)$$

Then the total reconstruction error function is written in the form:

$$\left\langle \left[\delta^2(t)\right]^k \right\rangle = \sigma^2(t) + \left[\varepsilon^k\left(t;\hat{m}^k,\left[\sigma^2\right]^k,\hat{R}^k\right)\right]^2. \quad (21)$$

The formulas (18) - (21) are more general in the description of the adaptive SRP. It is clear that the mentioned formulas (18) - (21) can be rewritten for many variants, when one or two parameters are unknown. Again we notice, when the number k of samples participated in the parameter estimation is increased, the result of the estimation tends to the true value. Then the additional error should be decreased to zero. Below we will investigate adaptive algorithms for the Markovian Gaussian process. In this case the number of samples involved in the reconstruction operation equals two. The algorithms for the simulation of random process realizations are not discussed here. We notice some parameters of simulated realization only: $m=0$, $\sigma^2=1$ and $\tau_c=1$ (τ_c is the covariance time).

6 The Mathematical Expectation is Unknown

Essential formulas for this variant are given in section 4. In Fig. 1 and Fig. 2 graphs show the modulus of the difference between two reconstruction functions: 1) the mathematical expectation $\tilde{m}(t)$ and 2) the estimated reconstruction function $m^k(t)$.

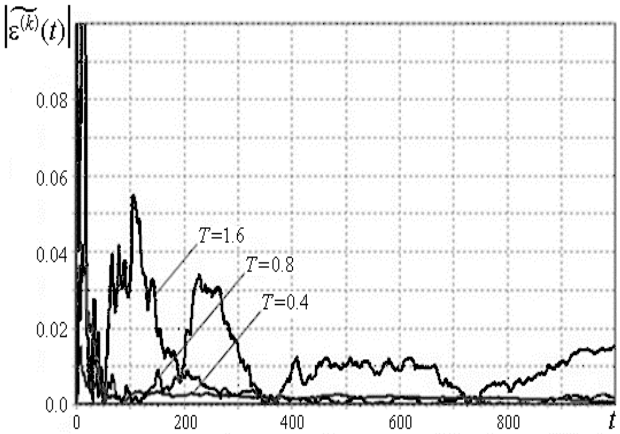

Fig. 1

For the sake of simplification, the calculations were fulfilled in the middle of the sampling intervals T. Curves are presented for different sampling intervals. In Fig 1 we have functions of time t. In Fig 2 the graphs have the argument number k.

One can notice that all curves in Fig. 1 – Fig. 3 illustrate a natural effect: when time t (or the step number k) increases, the additional error decreases.

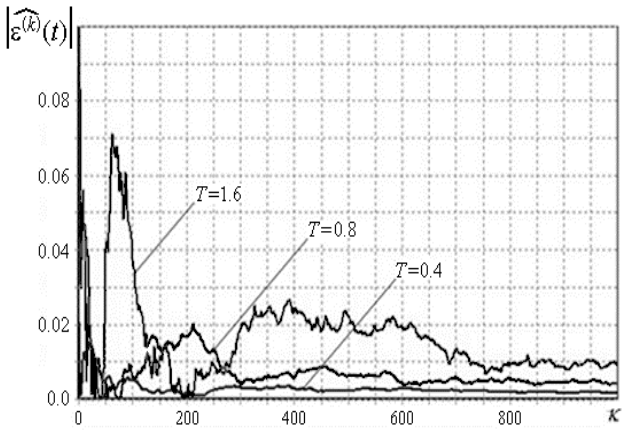

Fig. 2

But there is another very important point: the values of the additional error values are very tiny in comparison with a usual reconstruction error (the conditional variance). Actually maxima error in the middle of the sampling interval $t=T/2$ is characterized by the following values [7]:

$$\begin{aligned} &T=0.2, \sigma^2_{\max}(T/2)\approx 0.1;\\ &T=0.4, \sigma^2_{\max}(T/2)\approx 0.2;\\ &T=0.6, \sigma^2_{\max}(T/2)\approx 0.29;\\ &T=0.8, \sigma^2_{\max}(T/2)\approx 0.38. \end{aligned} \quad (22)$$

Meantime the additional error is described by values very small (see the ordinate scale in Fig. 3.

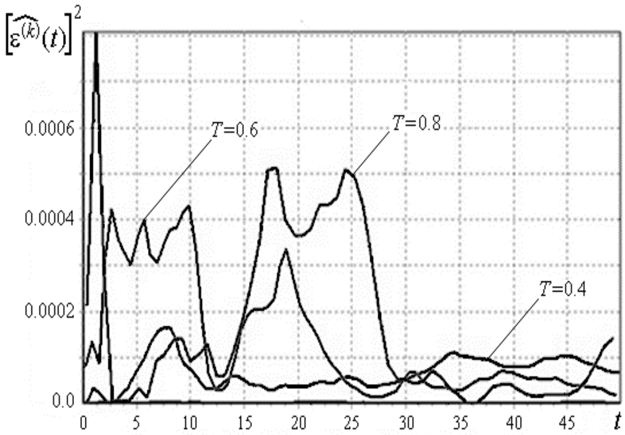

Fig. 3

7 The Variance is Unknown

In order to calculate the case with unknown variance, it is necessary to use the general formulas (18) – (21) from section 4 with some changes. Namely, the upper index k must be omitted in all designations except the variance $\left[\sigma^2\left(t\right)\right]^k$, because other parameters are known. We do not write corresponding expressions, but give the calculation results of the additional error $\left[\varepsilon^{(k)}\left(t\right)\right]$ (see Fig. 4).

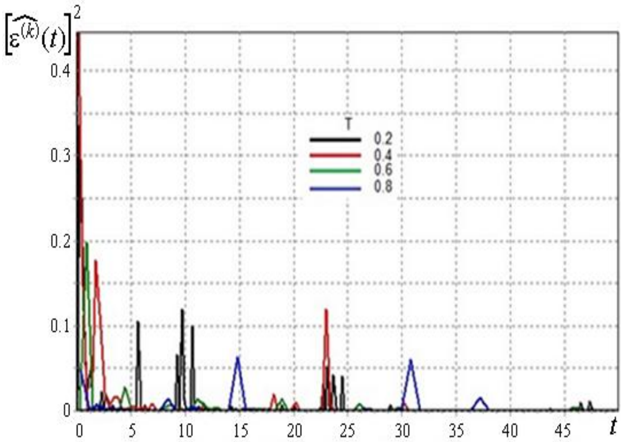

Fig. 4

In Fig. 4, there are some curves for different values of sampling interval T. The curves have been obtained on the values of errors in the middle of sampling intervals.

One can notice that all broken curves tend to zero when the analysis time increases. Once again, one can make the conclusion: the additional error (see the ordinate scale in Fig. 4) is very small with comparison of the usual error (see the ordinate scale in Fig. 4 and the values in (22)).

8 The Covariance Moment is Unknown

The formulas (18) - (21) can be concretized for the general case. Let us suppose that the type of covariance function is known. In our case this type is exponential. Then, it is sufficient to estimate the covariance moment between two random variables, divided by several sampling intervals. (This is the reason that we estimate the covariance moment, but not the covariance function.) Then in general in formulas (18) - (21) we have to keep the index k for the covariance moment $\hat{R}^k$ only, because other parameters must be known. Simulation results are shown in Fig. 5 and Fig. 6.

Let us consider Fig. 5. Here there are four curves. They illustrate the transition regimen of the estimation of the covariance for various sampling intervals, when time increases. In all cases, the results of the estimation tend to the exact value of the covariance moment for the sampling interval T under consideration. When the sampling interval is small, the covariance moment is large and vice versa.

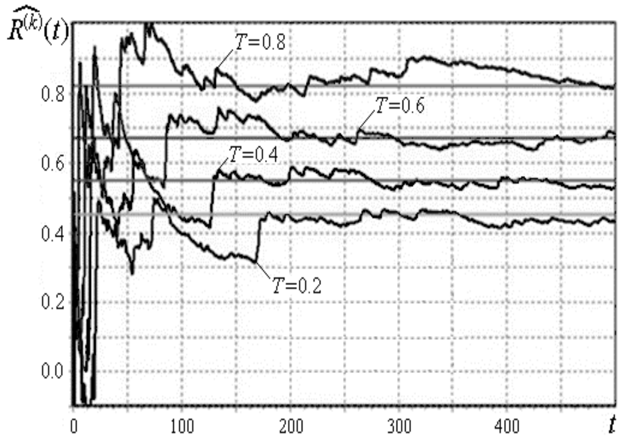

Fig. 5

In Fig. 6 graphs of total errors $\left\langle\left[\delta^2\left(t\right)\right]^k\right\rangle$ (see the expression (21)) are presented. But for the sake of simplification, instead of the function $\sigma^2\left(t\right)$ we chose the constant $\sigma^2\left(t\right)=\sigma^2\left(T/2\right)$. These values are marked by T=0.2, 0.4, 0.6 and 0.8. All curves show that the additional error tends to zero, when the observation time interval (or the number k) increases. The transition interval duration is not constant, because the sampling intervals T are different. Meantime the number of samples k within the transition interval duration is nearly equal for all curves.

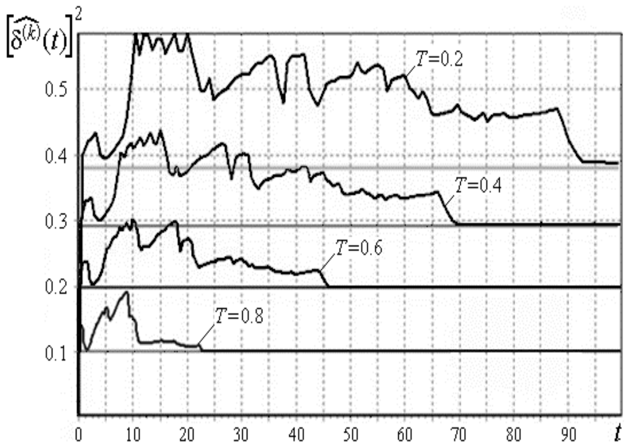

Fig. 6

Here we notice one specific feature of graphs in Fig. 2. In fact, all curves (especially during the initial part of transition interval) have values compared with the correspondent value $\sigma_{\max}^2\ t = \sigma^2\ T/2$.

There is not such effect in two above investigated variants: the mathematical expectation (section 6) and variance are unknown (section 7). This effect means that the information lack about the covariance moment provokes additional errors larger compared with the other variants, investigated above. Generally, it can be said that the adaptive algorithm for the covariance function (or particularly the covariance moment) plays a major role in adaptive algorithms SRP of Gaussian processes. This conclusion is valid for the SRP of non Markovian processes also.

9 Conclusions

The investigation method of adaptive algorithms in the description of SRP of realization of Gaussian processes is suggested. The general formulas are obtained. The method is illustrated by realizations of Markovian process. The practical recommendation is: among three unknown principal parameters the lack of the information about the covariance function plays the grand role in the adaptive SRP algorithms of Gaussian processes.

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