Direction-of-arrival estimation of a single distorted wavefront with time-variant amplitude

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Abstract: Direction-of-arrival (DOA) estimation algorithms in array processing applications have been developed under time-invariant wavefronts. In most applications this assumption is not realistic due to the nonhomogeneous propagation medium which can distort the wavefront received by the array. This paper extends the author's previous work on signal-to-noise ratio (SNR) estimation, in developing a novel approach for estimating the DOA of a single narrow-band amplitude-distorted wavefront received by an *arbitrary antenna array*. The distorted-amplitude wavefront is assumed to vary according to the first order autoregressive AR(1) model with unknown coefficients. An approximate maximum-likelihood-based (ML-based) approach to estimate the DOA parameter is developed in the high SNR scenario. Compared with the classical ML method that requires computationally prohibitive multi-dimensional search, the proposed approach obtains the DOA estimate by maximizing a new cost function with respect to a single DOA parameter derived using Markov property of the AR(1) process. Compact Cramér-Rao lower bound (CRB) expressions for DOA parameter are derived for different kinds of time-varying fading amplitudes. High and low SNR approximation expressions for the CRB are also derived, that enable the derivation of a number of CRB properties. Finally, simulation results show the performance of the proposed estimator and validate the theoretical analysis.

Key–Words: Direction-of-arrival (DOA) estimation, maximum likelihood (ML), Cramér Rao bound, Time-varying complex-valued AR(1) model.

1 Introduction

Direction-of-arrival (DOA) estimation of multiple signals using sensor arrays have been popular in radar, sonar and wireless communications for decades (e.g., [1–4]). Stochastic and deterministic CRBs derivations for the DOA parameter have been investigated repeatedly because the performance of several high-resolution DOA estimation methods are known to be comparable to these bounds under certain mild conditions (e.g., [5], [6]). The majority of DOA estimation methods (e.g., [7–9]) generally assume that the waveforms are known. In particular, the problem of estimating the DOA of a single source has been extensively studied with time-invariant channel (e.g., [10,11]). A low complexity, fast and explicit approximate ML algorithm has been developed in [10]. For constant-modulus signal, a ML DOA estimator was derived in [11], which utilizes the prior knowledge of the signal waveform. However, in many signal processing applications involving propagation media that are neither homogeneous nor isotropic. There are also applications where the antenna array may be in motion due to the variation in DOA and Doppler shift. Examples of such applications include sonar, radar and underwater communication systems. As a consequence, there are both random amplitude and phase errors occurring due to the wavefront distortions. Also, the correlation matrix of the received distorted-wavefront is not full rank. Thus, the high-resolution DOA estimation methods are not applicable. The problem of narrowband DOA estimation with imperfect coherent (randomly distorted) wavefronts was studied in [12–15]. However, the developed approaches often require a priori knowledge of the spatial coherence matrix [12, 13]. In [12], Paulraj and Kailath presented a subspace algorithm with known coherence variation. Gershman *et al.* [14] developed a procedure to jointly estimate the spatial coherence loss and the DOA's. A robust estimator was proposed in [15], based on a reduced statistic obtained from the subdiagonals of the covariance matrix of the uniform linear array (ULA) output.

This paper extends the author's previous work that was developed for SNR estimation [21], in developing a novel approach for estimating the DOA of a single narrow-band amplitude-distorted wavefront received by an arbitrary antenna array. This paper also gives detailed proofs of the main results (presented without justification) in [22] followed by a detailed discussion and original illustrations. The time-variant complex-valued amplitude of the wavefront are modeled as an AR(1) process, similar to [21, 23], but the AR(1) process parameters are assumed unknown. An approximate ML estimation procedure is derived for the high-SNR scenario, to simultaneously estimate both the DOA parameter and the time-variant amplitudes parameters. The estimation procedure requires only a single-dimensional parameter search with respect to the DOA parameter. A closed-form expressions of the CRB are derived with correlated and uncorrelated time-variant complex-valued channel amplitude. This bounds enable the derivation of several properties that describe the effect of the time-variant amplitude on DOA estimation.

The rest of this paper is organized as follows. Section 2 describes the signal model and AR(1) correlation model and formalizes the estimation problem. In Section 3, the ML estimator is derived for a high SNR approximation. In Section 4, exact and approximate closed-form expressions for the CRB of the DOA parameter alone are derived for fast amplitude fading. Closed-form expressions for the CRBs associated with slow and uncorrelated amplitude fading are derived in Section 5. Section 6 derives a variety of properties of the derived bounds. In Section 7, simulation results are presented to illustrate the results. Finally, section 8 concludes the paper.

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. $\mathbf{I}$ is the identity matrix. Vectors are by default in column orientation, while T, H and $*$ stand for transpose, conjugate transpose and conjugate, respectively. $\mathrm{E}(.)$, $\mathrm{Tr}(.)$, $\|.\|$, and $\otimes$ are the expectation, trace, norm operators, and Kronecker product, respectively. $(\mathbf{A})_{i,j}$ denotes the (i,j)th element of the matrix $\mathbf{A}$.

2 Signal Model and problem formulation

Let an arbitrary array of M sensors receive a narrow-band distorted wavefront from an unknown impinging angle θ. The tth observed sample $y_k(t)$ of the kth sensor in the array can be written as (e.g., [14, 16])

$$y_k(t) = h_k(t)a_k(\theta) + n_k(t), \tag{1}$$

for $t = 0, \dots, N-1$ and $k = 1, \dots, M$, where $a_k(\theta) = e^{i\tau_k(\theta)}$, with $\tau_k(\theta)$ being the DOA-dependent time needed by the wavefront to travel from the first to the kth sensor, $h_k(t) \in \mathbb{C}$ is the amplitude distortion, and it is assumed to be zero-mean circular complex Gaussian with unknown variance σ_h^2 and $n_k(t)$ is the noise term. Since the size of the sensor array is relatively small to the propagation distance (far-field scenario), it is reasonable to assume that the complex-amplitudes are space-invariant (i.e., $h_k(t) = h(t)$) and time-variant according to a complex-valued AR(1) model. Hence, in this ideal case, the $M \times 1$ vector of signals received by the array can be represented as

$$\begin{aligned}\mathbf{y}(t) &\stackrel{\text{def}}{=} (y_1(t), \dots, y_M(t))^T \\ &= h(t)\mathbf{a}(\theta) + \mathbf{n}(t),\ t = 0, \dots, N-1, \end{aligned} \tag{2}$$

where $\mathbf{a}(\theta) \stackrel{\text{def}}{=} (e^{i\tau_1(\theta)}, \dots, e^{i\tau_M(\theta)})$ is the steering vector parameterized by the unknown scalar DOA parameter θ. We suppose $\|\mathbf{a}(\theta)\|^2 = M$. The M-variate additive noise vectors $\{\mathbf{n}(t)\}_{t=0}^{N-1}$ are assumed to be i.i.d. zero-mean complex circular Gaussian with covariance matrix $\mathrm{E}(\mathbf{n}(t)\mathbf{n}^H(t)) = \sigma_n^2\mathbf{I}$. The signal-to-noise ratio (SNR) is defined as $\rho \stackrel{\text{def}}{=} \frac{\sigma_h^2}{\sigma_n^2}$. The amplitudes $h(t)$ are modelled by the complex-valued AR(1) process as

$$h(t) = \gamma h(t-1) + \sqrt{1-\gamma^2}e(t), \tag{3}$$

where $e(t) \sim \mathcal{N}(0, \sigma_h^2)$ is the additive driving noise and γ is the AR(1) correlation parameter[1] assumed to be unknown and defined as the normalized Jakes correlation at lag one (e.g., [20, 21, 23, 24]) $\gamma \stackrel{\text{def}}{=} \mathrm{J}_0(2\pi f_d T)$ where J_0 is the zeroth-order Bessel function of the first kind and f_dT is the normalized Doppler frequency of the correlation channel. The amplitude at time t is constrained to follow a sequence from a known initial state, say $h(0)$:

$$h(t) = \gamma^t h(0) + \sqrt{1-\gamma^2}\sum_{k=0}^{t-1}\gamma^k e(t-k). \tag{4}$$

The correlation over m signalling intervals is given by

$$R_h^{\mathrm{AR}}(m) = \mathrm{E}(h(t)h^*(t+m)) = \sigma_h^2\gamma^{|m|}.$$

Consequently, the covariance matrix of $\mathbf{h} \stackrel{\text{def}}{=} (h(0), \dots, h(N-1))^T$ is a symmetric Toeplitz matrix and can be written as

$$\mathbf{R}_h = \sigma_h^2 \begin{pmatrix} 1 & \gamma & \gamma^2 & \dots & \gamma^{N-1} \\ \gamma & 1 & \gamma & \dots & \gamma^{N-2} \\ \gamma^2 & \gamma & 1 & \dots & \gamma^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma^{N-1} & \gamma^{N-2} & \gamma^{N-3} & \ddots & 1 \end{pmatrix}. \tag{5}$$

[1] This parameter is also called the coherence loss parameter [15].

Note that the matrix (5) becomes a diagonal matrix for $\gamma = 0$, and so the channel amplitude becomes an uncorrelated process and for $\gamma = 1$ the channel amplitude is simply a realization of a single random variable (slowly varying complex amplitude).

Collecting the samples of the received signal to form a vector $\mathbf{y} \stackrel{\text{def}}{=} (\mathbf{y}(0)^T, \ldots, \mathbf{y}(N-1)^T)^T$ yields the following model

$$\mathbf{y} = \mathbf{A}\mathbf{h} + \mathbf{n}, \tag{6}$$

where $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{I} \otimes \mathbf{a}(\theta)$ and $\mathbf{n} \stackrel{\text{def}}{=} \big(\mathbf{n}(0)^T, \ldots, \mathbf{n}(N-1)^T\big)^T$ is a $NM \times 1$ noise vector with covariance matrix $\sigma_n^2\mathbf{I}$. The vector $\mathbf{y}$ is a zero-mean complex Gaussian random vector, with correlation matrix given by

$$\mathbf{R}_y \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{y}\mathbf{y}^H) = \mathbf{A}\mathbf{R}_h\mathbf{A}^H + \sigma_n^2\mathbf{I}. \tag{7}$$

The probability density function (pdf) of $\mathbf{y}$ is given by:

$$p(\mathbf{y};\boldsymbol{\alpha}) = \frac{1}{\pi^{NM}\det(\mathbf{R}_y)} e^{-\mathbf{y}^H\mathbf{R}_y^{-1}\mathbf{y}}, \tag{8}$$

where $\boldsymbol{\alpha} = (\theta, \boldsymbol{\alpha}_n^T)^T$ is an unknown parameter vector depending on the θ parameter and, a vector of nuisance parameters $\boldsymbol{\alpha}_n \stackrel{\text{def}}{=} (\sigma_n^2, \sigma_h^2, \gamma)^T$.

The estimation problem can now be formulated as follows: Given the received signal $\mathbf{y}$ with pdf in (8), estimate the parameter of interest θ in the presence of a nuisance parameter $\boldsymbol{\alpha}_n$.

Extremely slow fading amplitude can be seen as a degenerate case of a completely correlated process $h(t)$, i.e., $R_h(m) = \sigma_h^2$. The observation vector for the slow-fading channel model can be written as

$$\mathbf{y}_{sla} = h\mathbf{a}(\theta) + \mathbf{n},\ h \sim \mathcal{N}(0, \sigma_h^2) \tag{9}$$

with covariance matrix $\mathbf{C}_y = \sigma_h^2\mathbf{a}(\theta)\mathbf{a}(\theta)^H + \sigma_n^2\mathbf{I}$ and nuisance parameter vector $\boldsymbol{\alpha}_n^{sla} \stackrel{\text{def}}{=} (\sigma_n^2, \sigma_h^2)^T$. Subsequently, the results for the fast-fading amplitude will be compared with that of the slow-fading amplitude.

3 ML DOA estimator

3.1 Outline of Approach

The direct maximization of the likelihood function (8) with respect to the unknown parameter $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\theta, \boldsymbol{\alpha}_n^T)^T$ is a difficult task. The simple approach for deriving the estimates of the parameter vector $\boldsymbol{\alpha}$ is to concentrate (8) with respect to the nuisance parameters, and to perform a search on the DOA parameter and the nuisance parameters that cannot be concentrated. This approach has been investigated in [18,19], which concentrate the ML estimation problem with respect to the signal covariance matrix elements and the noise power. However, due to the particular structure of the current model (2), such a solution can not compresses (8) with respect to all nuisance parameters[2] (i.e., $\boldsymbol{\alpha}_n$). Thus, this section presents another approach that exploits the Markov property[3] of the AR(1) process and applying the chain rule for expressing the pdf of $\mathbf{y}$ in terms of the product of conditional pdfs. This section shows also that it is possible to reduce the optimization problem, under a high SNR approximation, to a *single-parameter search* with respect to the DOA parameter θ.

The concentration procedure can be performed as

$$\begin{aligned}
\hat{\theta}_{ML} &= \arg\max_{\theta} F(\theta;\mathbf{y}),\\
F(\theta;\mathbf{y}) &= \max_{\boldsymbol{\alpha}_n} L(\boldsymbol{\alpha}_n,\theta;\mathbf{y}),\\
L(\boldsymbol{\alpha}_n,\theta;\mathbf{y}) &\stackrel{\text{def}}{=} \ln(p(\mathbf{y};\boldsymbol{\alpha})) \qquad (10)\\
&= \ln(NM\pi) - \ln\det(\mathbf{R}_y) - \mathbf{y}^H\mathbf{R}_y^{-1}\mathbf{y},
\end{aligned}$$

where $\mathbf{R}_y = \mathbf{A}\mathbf{R}_h\mathbf{A}^H + \sigma_n^2\mathbf{I}$, and $\mathbf{R}_h$ is defined in (5). The next subsection concentrates on the derivation of $F(\theta;\mathbf{y})$ to estimate DOA, and provides explicit expressions for the ML estimates of the nuisance parameters under a high SNR approximation.

3.2 Evaluating the conditional PDF

Using the Markov property of the AR(1) process, an equivalent negative log-likelihood function to (10) is proved in appendix A and is given by (after dropping the constant term)

$$\begin{aligned}
L(\boldsymbol{\alpha}_n,\theta;\mathbf{y}) &= N\ln(\sigma_n^{2(M-1)}\sigma_h^2) + N\ln\Big(1 + \frac{\sigma_n^2}{M\sigma_h^2}\Big)\\
&\quad + (N-1)\ln\Big(1 - \gamma^2\frac{1}{(1+\frac{\sigma_n^2}{M\sigma_h^2})^2}\Big) \qquad (11)\\
&\quad + \mathbf{y}(0)^H\mathbf{C}_y^{-1}\mathbf{y}(0) + \sum_{n=1}^{N-1}\bar{\mathbf{y}}(n)^H\mathbf{C}^{-1}\bar{\mathbf{y}}(n),
\end{aligned}$$

where $\bar{\mathbf{y}}(n) \stackrel{\text{def}}{=} \mathbf{y}(n) - \frac{\gamma\sigma_h^2}{M\sigma_h^2+\sigma_n^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1)$, $\mathbf{C}_y \stackrel{\text{def}}{=} \sigma_h^2\mathbf{a}(\theta)\mathbf{a}^H(\theta) + \sigma_n^2\mathbf{I}$ and $\mathbf{C} \stackrel{\text{def}}{=} \sigma_h^2\Big(1 - \frac{\gamma^2 M\sigma_h^2}{M\sigma_h^2+\sigma_n^2}\Big)\mathbf{a}(\theta)\mathbf{a}^H(\theta) + \sigma_n^2\mathbf{I}$.

Remark 1 *Note that for* $\gamma = 0$*, the log-likelihood function* (11) *is reduced to the following stochastic*

[2] We were able to reduce the search to a two-dimensional search in θ and γ.

[3] A stochastic process $\{x_t\}$ is markovian if $p(x_{t+1}|x_t, x_{t-1}, \ldots, x_{t-k}) = p(x_{t+1}|x_t),\ \forall k$

ML log-likelihood function associated with the DOA estimation of a single source:

$$L(\bar{\boldsymbol{\alpha}}_n, \theta; \mathbf{y}) = N \ln(\sigma_n^{2(M-1)}\sigma_h^2) + N \ln(1 + \frac{\sigma_n^2}{M\sigma_h^2}) + \sum_{n=0}^{N-1} \mathbf{y}(n)^H \mathbf{C}_y^{-1} \mathbf{y}(n).$$

Here, $\bar{\boldsymbol{\alpha}}_n \stackrel{\text{def}}{=} (\sigma_h^2, \sigma_n^2)^T$*, and the matrix* $\mathbf{R}_h$ *embedded in the covariance matrix* $\mathbf{R}_y$ *can be seen as the covariance matrix of the signal source. The ML estimates of the DOA parameter* θ*, and nuisance parameters* σ_h^2 *and* σ_n^2 *can be obtained in separable*[4] *form as shown in (e.g., [18,19]) for the general case of multiple signal sources.*

Note that the maximization of (11) is significantly complicated by the presence of the nuisance parameter γ. However, since the objective of this section is to find a simple method of estimation, a high-SNR approximation is restored in the following subsection.

3.3 High-SNR ML estimator

Using (42), and for high SNR (σ_n^2 is very small), $\mathbf{C}_y^{-1}$ can be approximated as

$$\mathbf{C}_y^{-1} \approx \frac{1}{\sigma_n^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\sigma_h^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta).$$

Because $\mathbf{C}$ has same structure as $\mathbf{C}_y$, its inverse can also be approximated as

$$\mathbf{C}^{-1} \approx \frac{1}{\sigma_n^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\sigma_h^2(1-\gamma^2)}\mathbf{a}(\theta)\mathbf{a}^H(\theta), \gamma \neq 1.$$

Hence, for high SNR, Eq. (11) can be simplified as

$$\begin{aligned} L'(\boldsymbol{\alpha}_n, \theta; \mathbf{y}) &= N \ln(\sigma_n^{2(M-1)}\sigma_h^2) + (N-1)\ln(1-\gamma^2) \\ &+ \mathbf{y}(0)^H \left(\frac{1}{\sigma_n^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\sigma_h^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\right)\mathbf{y}(0) \quad (12) \\ &+ \sum_{n=1}^{N-1} \tilde{\mathbf{y}}(n)^H \tilde{\mathbf{C}} \tilde{\mathbf{y}}(n), \ \gamma \neq 1, \end{aligned}$$

where $\tilde{\mathbf{y}}(n) \stackrel{\text{def}}{=} \mathbf{y}(n) - \frac{\gamma}{M}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1)$, $\Pi^{\perp}_{\mathbf{a}(\theta)} \stackrel{\text{def}}{=} \mathbf{I} - \frac{1}{M}\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ and $\tilde{\mathbf{C}} \stackrel{\text{def}}{=} \frac{1}{\sigma_n^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\sigma_h^2(1-\gamma^2)}\mathbf{a}(\theta)\mathbf{a}^H(\theta)$.

The following main result proved in appendix B, shows that it is possible to reduce the optimization problem, under a high SNR approximation, to a single-parameter search with respect to the DOA parameter θ.

[4]The DOA parameter can be obtained by maximizing a function of only the DOA parameter.

Result 1 *For high SNR environment and* $\gamma \neq 1$*, the joint ML estimates of the parameter vector* $\boldsymbol{\alpha}$ *that maximize the log-likelihood function* (11) *are given by the following:*
$\hat{\theta}_{ML}$ *is obtained by the maximizing with respect to* θ

$$\begin{aligned} &F(\theta; \mathbf{y}) = \\ &- \Big(N \ln(\hat{\sigma}_{n,ML}^{2(M-1)}\hat{\sigma}_{h,ML}^2) + (N-1)\ln(1-\hat{\gamma}_{ML}^2) \\ &+ \mathbf{y}(0)^H \left(\frac{1}{\hat{\sigma}_{n,ML}^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\hat{\sigma}_{h,ML}^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\right)\mathbf{y}(0) \\ &+ \sum_{n=1}^{N-1} \tilde{\mathbf{y}}(n)^H \tilde{\mathbf{C}} \tilde{\mathbf{y}}(n)\Big), \quad (13) \end{aligned}$$

where $\tilde{\mathbf{y}}(n) \stackrel{\text{def}}{=} \mathbf{y}(n) - \frac{\hat{\gamma}_{ML}}{M}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1)$ *and* $\tilde{\mathbf{C}} \stackrel{\text{def}}{=} \frac{1}{\hat{\sigma}_{n,ML}^2}\Pi^{\perp}_{\mathbf{a}(\theta)} + \frac{1}{M^2\hat{\sigma}_{h,ML}^2(1-\hat{\gamma}_{ML}^2)}\mathbf{a}(\theta)\mathbf{a}^H(\theta)$ *and where* $\hat{\sigma}_{h,ML}^2$*,* $\hat{\sigma}_{n,ML}^2$ *and* $\hat{\gamma}_{ML}$ *are the estimates of the nuisance parameters given by*

$$\hat{\gamma}_{ML} = -\frac{k_{2,y}(\theta)}{2k_{4,y}(\theta)}, \quad (14)$$

$$\hat{\sigma}_{h,ML}^2 = \frac{1}{N}\left(k_{3,y}(\theta) + \frac{1}{1-\hat{\gamma}_{ML}^2}(-\hat{\gamma}_{ML}k_{2,y}(\theta) + \hat{\gamma}_{ML}^2 k_{1,y}(\theta))\right), \quad (15)$$

$$\hat{\sigma}_{n,ML}^2 = \frac{1}{N(M-1)}\sum_{n=0}^{N-1} \mathbf{y}(n)^H \Pi^{\perp}_{\mathbf{a}(\theta)} \mathbf{y}(n), \quad (16)$$

where the DOA-dependent coefficients $k_{l,y}(\theta)$*,* $l = 1, ..., 4$*, are given by*

$$\begin{aligned} k_{1,y}(\theta) &\stackrel{\text{def}}{=} \frac{1}{M^2}\left(\sum_{n=1}^{N-1}(\mathbf{y}(n)^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{y}(n) + \mathbf{y}(n-1)^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{y}(n-1))\right), \\ k_{2,y}(\theta) &\stackrel{\text{def}}{=} \frac{1}{M^2}\left(\sum_{n=1}^{N-1}(\mathbf{y}(n)^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{y}(n-1) + \mathbf{y}(n-1)^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{y}(n))\right), \\ k_{3,y}(\theta) &\stackrel{\text{def}}{=} \frac{1}{M^2}\sum_{n=0}^{N-1}\mathbf{y}(n)^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{y}(n), \\ k_{4,y}(\theta) &\stackrel{\text{def}}{=} k_{3,y}(\theta) - k_{1,y}(\theta). \end{aligned}$$

The overall estimation procedure can be summarized as follows. For each value of θ in the search domain, the ML estimates of σ_h^2, γ, and σ_n^2 are given by (15), (14) and (16), respectively. Substituting the estimates of the nuisance parameters into (12) yields

(13). The ML estimate of θ is obtained by maximizing (13). Thus, for high SNR, the nuisance parameters are given in closed-form expressions that depend on the DOA parameter, reducing the search to a *single-parameter search* on the DOA parameter only.

Note that the ML approach only requires maximizing (13) with respect to a scalar θ, which can be efficiently implemented using derivative-free uphill search methods such as the Nelder-Mead algorithm[5] [25].

From (14) and (16), using the high SNR condition, we get

$$\begin{aligned} \hat{\gamma}_{ML} &= \frac{c_{n,1}(\theta)}{c_{n,2}(\theta)} \\ &\overset{N\to\infty}{\longrightarrow} \frac{(N-1)M^2\sigma_h^2 \mathrm{J}_0(2\pi f_d T)}{(N-2)(M^2\sigma_h^2 + M\sigma_n^2)} \\ &\approx \quad \mathrm{J}_0(2\pi f_d T) \qquad (17) \\ \hat{\sigma}^2_{n,ML} &\overset{N\to\infty}{\longrightarrow} \sigma_n^2. \qquad (18) \end{aligned}$$

$c_{n,1}(\theta) \overset{\text{def}}{=} \sum_{n=1}^{N-1}(\mathbf{y}(n)^H\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1) + \mathbf{y}(n-1)^H\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n))$ and $c_{n,2}(\theta) \overset{\text{def}}{=} \sum_{n=2}^{N-1}\mathbf{y}(n-1)^H\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1)$.

Similarly, from (15) using (17) and under the high-SNR approximation, we get after some easy manipulations

$$\hat{\sigma}^2_{h,ML} \overset{N\to\infty}{\longrightarrow} \sigma_h^2.$$

Consequently, $\hat{\gamma}_{ML}$, $\hat{\sigma}^2_{h,ML}$ and $\hat{\sigma}^2_{n,ML}$ are consistent estimators of γ, σ_h^2 and σ_n^2, respectively at high SNR.

4 Exact and approximation forms of CRB

This section presents various exact and approximate forms of the CRB for the DOA parameter in fast and slow time-variant channel amplitude.

4.1 General Expression

Since the data vector $\mathbf{y}$ is zero mean, complex, circular, and Gaussian with covariance matrix $\mathbf{R}_y$, which is dependent on the parameter vector $\boldsymbol{\alpha}$, the CRB for $\boldsymbol{\alpha}$ can be expressed as [26, rel. B.3.25]:

$$\mathrm{CRB}(\boldsymbol{\alpha}) = (\mathbf{I}_\alpha)^{-1},$$

where $\mathbf{I}_\alpha$ is the Fisher information matrix (FIM) given by

$$(\mathbf{I}_\alpha)_{k,l} \overset{\text{def}}{=} \mathrm{Tr}\left(\mathbf{R}_y^{-1}\frac{\partial \mathbf{R}_y}{\partial \alpha_k}\mathbf{R}_y^{-1}\frac{\partial \mathbf{R}_y}{\partial \alpha_l}\right),\ k,l = 1,\ldots,4. \qquad (19)$$

[5]The Nelder-Mead algorithm has already been incorporated in the function “fminsearch” in MATLAB®.

The expression of the CRB for the DOA parameter alone proved in appendix C, is summarized by the following result.

Result 2 *For arbitrary array geometries, the expression of the CRB for the DOA parameter (i.e., θ) is decoupled from the nuisance (distortion) parameters (i.e., $\boldsymbol{\alpha}_n$) in the presence of time-variant complex-valued channel amplitude, and is given by:*

$$\mathrm{CRB}(\theta) = \mathrm{CRB}_0^{\mathrm{DA}}(\theta)\frac{N\sigma_h^2}{\mathrm{Tr}\left(\mathbf{R}_h^2\left(\mathbf{R}_h + \frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1}\right)}, \qquad (20)$$

where $\mathrm{CRB}_0^{\mathrm{DA}}(\theta) = \frac{1}{N\rho}\frac{1}{\alpha}$ denotes the data-aided(DA)[6] CRB derived in [17] for time-invariant (constant) channel amplitude and where α is the purely geometrical factor[7] $2\mathbf{a}'^H(\theta)\Pi^{\perp}_{\mathbf{a}(\theta)}\mathbf{a}'(\theta)$ with $\Pi^{\perp}_{\mathbf{a}(\theta)} \overset{\text{def}}{=} \mathbf{I} - \mathbf{a}(\theta)\mathbf{a}^H(\theta)/M$ and $\mathbf{a}'(\theta) \overset{\text{def}}{=} \frac{\partial \mathbf{a}(\theta)}{\partial\theta}$.

Remark 2 *It is important to remark that the structure of the general CRB expression* (20) *is that of the* $\mathrm{CRB}_0^{\mathrm{DA}}$ *increased by a factor* $\frac{N\sigma_h^2}{\mathrm{Tr}\left(\mathbf{R}_h^2\left(\mathbf{R}_h+\frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1}\right)}$ *which is depends on both SNR and* $\mathbf{R}_h$.

Remark 3 *It is shown in appendix C that the DOA parameter is decoupled from the nuisance (distortion) parameters in the FIM. Consequently, the CRB* (20) *remains valid for any type of fading correlation (e.g., Jakes' [20] and AR correlation channel models) when the amplitude fading process is complex normal and of zero mean.*

4.2 Approximate expressions for CRB

To get more insights on the CRB, the following subsection gives approximate expressions for the CRB (20) in the high and low SNR regimes that enable the derivation of the properties below.
For high and low SNR cases, we have

$$\begin{aligned} \left(\mathbf{R}_h + \frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1} &\approx \mathbf{R}_h^{-1} \text{ for high SNR} \\ \left(\mathbf{R}_h + \frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1} &\approx \frac{M}{\sigma_n^2}\mathbf{I} \text{ for low SNR,} \end{aligned}$$

[6]In the DA case, the transmitted data symbols are assumed to be perfectly known.

[7]The parameter α is equal the following values $\alpha_{\mathrm{ULA}} = \pi^2\frac{M(M-1)}{6}\cos^2(\theta)$ [resp. $\alpha_{\mathrm{UCA}} = \frac{M\pi^2}{4\sin^2\pi/M}$] for uniform linear [resp. uniform circular] array.

hence, the channel-dependent term of the denominator of Eq. (20) can be approximated as:

$$\mathrm{Tr}\left(\mathbf{R}_h^2\left(\mathbf{R}_h+\frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1}\right) \approx N\sigma_h^2 \text{ for high SNR}$$
$$\mathrm{Tr}\left(\mathbf{R}_h^2\left(\mathbf{R}_h+\frac{\sigma_n^2}{M}\mathbf{I}\right)^{-1}\right) \approx \frac{M}{\sigma_n^2}\mathrm{Tr}(\mathbf{R}_h^2) \text{ for low SNR.}$$

Consequently, the expressions of the CRB for DOA alone for high and low SNR cases can be written as:

$$\mathrm{CRB}^{\mathrm{high}}(\theta) = \mathrm{CRB}_0^{\mathrm{DA}}(\theta) \text{ for high SNR,} \quad (21)$$
$$\mathrm{CRB}^{\mathrm{low}}(\theta) = \frac{1}{\rho^2\beta}\frac{1}{M\alpha} \text{ for low SNR,} \quad (22)$$

where the channel-dependent parameter β is given by $\frac{1}{\sigma_h^4}\mathrm{Tr}(\mathbf{R}_h^2)$. It is important to note that the CRB given by (21) is identical to the DA CRB derived in [17] with time-invariant channel amplitude.

5 Uncorrelated and slowly varying amplitudes cases

In the special cases of slow-varying amplitudes $(i.e., \gamma = 1$ and $\mathbf{R}_h = \sigma_h^2\mathbf{1}\mathbf{1}^T)$ and uncorrelated time-varying amplitudes $(i.e., \gamma = 0$, and $\mathbf{R}_h = \sigma_h^2\mathbf{I})$, result 2 can be extended to the following result

Result 3 *The CRB for DOA alone over slow and uncorrelated amplitude fading are given by*[8]

$$\mathrm{CRB}^{\mathrm{Slow}}(\theta) = \frac{1}{N}\left(\frac{1}{\alpha}\left[\frac{1}{\rho}+\frac{1}{MN\rho^2}\right]\right) \quad (23)$$
$$\mathrm{CRB}^{\mathrm{Uncor}}(\theta) = \frac{1}{N}\left(\frac{1}{\alpha}\left[\frac{1}{\rho}+\frac{1}{M\rho^2}\right]\right). \quad (24)$$

Note that the bound (24) is the conventional stochastic CRB for DOA alone of a single source derived in [6] under the circular complex Gaussian distribution. From (23) and (24), we obtain

$$\begin{aligned} \mathrm{CRB}^{\mathrm{Uncor}}(\theta) &\geq \mathrm{CRB}^{\mathrm{Slow}}(\theta) \text{ for all SNR} \\ \mathrm{CRB}^{\mathrm{Uncor}}(\theta) &\approx \mathrm{CRB}^{\mathrm{Slow}}(\theta) \\ &\approx \mathrm{CRB}_0^{\mathrm{DA}}(\theta) \text{ for high SNR} \quad (25) \\ \mathrm{CRB}^{\mathrm{Uncor}}(\theta) &\approx N\,\mathrm{CRB}^{\mathrm{Slow}}(\theta) \\ &\approx \frac{1}{N\rho^2}\frac{1}{M\alpha} \text{ for low SNR} \quad (26) \end{aligned}$$

[8]Here the superscripts Slow and Uncor of $\mathrm{CRB}^{\mathrm{Slow}}(\rho)$ and $\mathrm{CRB}^{\mathrm{Uncor}}(\rho)$ refer slow and uncorrelated channel fading respectively.

6 CRB properties

This section presents properties of the bound $\mathrm{CRB}(\theta)$ which are directly derived from the results of the previous section. It shows how the CRB depends on the key parameters such as SNR and channel parameter γ.

Property 1 *For high SNR, the CRBs for DOA alone associated with fast, slow and uncorrected time-variant channel amplitude are identical to the CRB for DOA alone associated with time-invariant (constant) amplitude, which is approximately inversely proportional to SNR and does not depend on the parameter of the channel.*

Proof: The proof follows from Eqs. (21) and (25). Note that for a given correlation model, the CRBs for DOA alone associated with slowly, rapidly and uncorrelated time-variant amplitudes are identical for high SNR accordingly to Eqs. (25) and (21).

Property 2 *For low SNR, the CRBs for DOA alone associated with fast, slow and uncorrected time-variant channel amplitude are approximately inversely proportional to ρ^2 (decreasing rapidly with SNR).*

Proof: The proof follows from Eqs. (26) and (22). Note that the parameter β in (22) is a monotone decreasing function of f_dT as illustrated in Fig. 1.

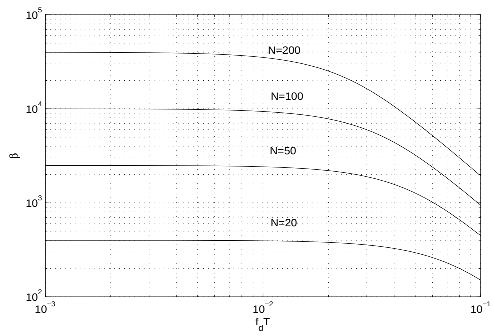

Fig.1 The channel-dependent parameter β for the AR(1) correlation model versus f_dT for different values of N.

It can be seen from this figure that β decreases rapidly except for slowly time-varying amplitudes (i.e., $\gamma \approx 1$). As the CRB (22) approximately inversely proportional to β, we obtain the following property

Property 3 *For low SNR, the CRB for DOA alone associated with the fast amplitude fading process is a monotonically decreasing function of the channel correlation parameter γ which varies from uncorrelated fading bound ($\gamma = 0$) to the slow fading bound ($\gamma = 1$).*

7 Simulation results

The purpose of this section is to illustrate the behavior of the derived CRB for DOA alone and the performance of the derived estimator.

Assume that a single narrowband source signal impinges on a uniform linear array (ULA) of $M = 6$ sensors separated by a half-wavelength for which $\mathbf{a} = \left(1, e^{i\theta}, \ldots, e^{i(M-1)\theta}\right)$, where $\theta = \pi \sin \alpha$, with α being the DOA relative to the normal of array broadside. The channel is simulated according to a AR(1) correlation model $[20, 21, 23]$ with doppler-time product of $f_d T$. In our simulations, 1000 Monte Carlo simulations were run to estimate the mean-square error (MSE) of the estimates.

We begin with Fig.2, which compares $\mathrm{CRB}^{\mathrm{Slow}}(\theta)$ (23), $\mathrm{CRB}^{\mathrm{Uncor}}(\theta)$ (24) and the exact CRB (20) versus SNR for two values of $f_d T$. It can be seen from this figure that all these bounds are identical except for low SNR where the fast amplitude fading bound decreasing from the CRB for uncorrelated amplitude fading (i.e.,$\gamma = 1$) to the CRB for slow amplitude fading (i.e.,$\gamma = 0$) as predicted by the Properties 1 and 3.

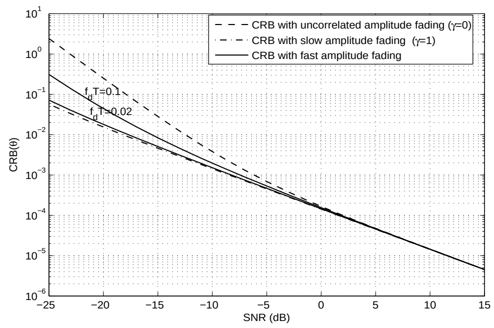

Fig.2 Exact CRB on DOA estimation with time-variant amplitude fading for two values of $f_d T$, $\mathrm{CRB}^{\mathrm{Slow}}(\theta)$ and $\mathrm{CRB}^{\mathrm{Uncor}}(\theta)$ versus SNR with $N = 200$.

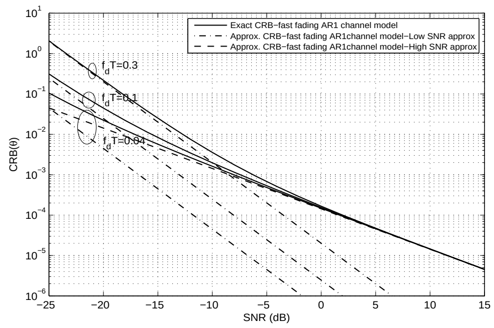

Fig.3 Exact CRB and its approximations versus SNR for three values of $f_d T$ with $N = 200$.

Fig.3 exhibits the domain of validity of the low and high-SNR approximations of the CRB given by Eqs. (22) and (21), respectively. It can be seen that the domain of validity depends on the values of $f_d T$, where for low SNR the exact CRB equals to its low approximation bound for a large SNR range except for small values of $f_d T$ (i.e., for slowly time-varying amplitude fading). At higher SNR, however, the approximate CRB does not depend on $f_d T$ which is identical to its exact bound for large SNR range as predicted by Property 1.

Fig.4 presents the dependence of the CRB for DOA alone on the time-variant AR(1) correlation model for low SNR throughout the Doppler-time product $f_d T$ for different values of N. We observe from this figure that as the Doppler-time product $f_d T$ increases, the CRBs remain quite constant up to Doppler-time product value of 0.0035, for which these bounds are identical to the CRB associated to the slow amplitude fading. We also see that the bounds increase when the time-Doppler product increases as predicted by Property 3.

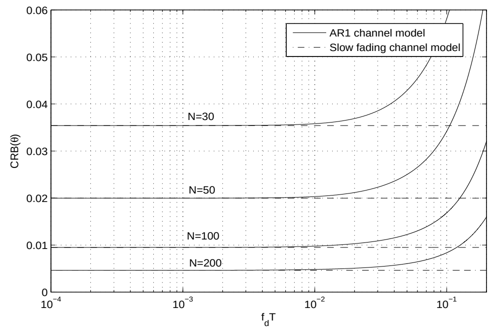

Fig.4 Exact $\mathrm{CRB}(\theta)$ for the time-varying AR(1) correlation model and $\mathrm{CRB}^{\mathrm{Slow}}(\theta)$ versus $f_d T$ with SNR$= -15dB$.

Fig.5 illustrates the Result 1 by comparing the exact CRB (20) and the minimum mean square error (MSE) of DOA estimate given by the asymptotic high-SNR ML estimator for the time-variant amplitude fading versus SNR. From this figure, we observe a good agreement between the derived CRB and the estimated MSE for high SNR. On the other hand, the asymptotic ML estimator still gives a valid estimate of DOA parameter for small values of N and for low SNR values.

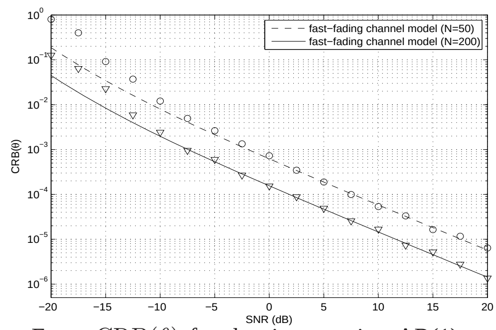

Fig.5 Exact $\mathrm{CRB}(\theta)$ for the time-varying AR(1) correlation model and estimated MSE $\mathrm{E}(\hat{\theta}_{ML} - \theta)^2$ given by the ML estimator versus SNR for two values of N with $f_d T = 0.01$.

In addition to the properties derived in section 6, Fig.6 presents the behavior of the CRB versus the number of observation N. It is seen that the CRB decreases as N increases. Observe that the MSE performance of the estimate reaches the CRB for a large range of SNR. This figure shows also that the derived high-SNR ML estimator continues to provide a good estimate for low SNR.

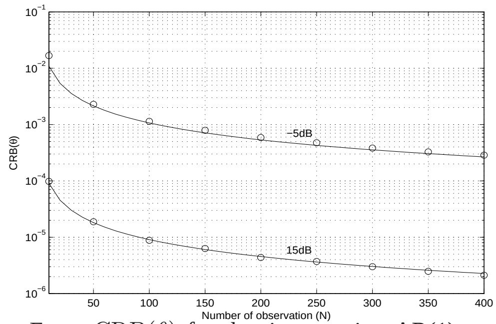

Fig.6 Exact $\mathrm{CRB}(\theta)$ for the time-varying AR(1) correlation model and estimated MSE $\mathrm{E}(\hat{\theta}_{ML}-\theta)^2$ given by the ML estimator versus N for two values of SNR with $f_dT = 0.1$.

8 Conclusion

This paper studies the direction-of-arrival (DOA) estimation problem of a single narrow-band source distorted by a time-varying complex AR(1) channel amplitude. A high signal-to-noise-ratio maximum likelihood (ML) estimator is proposed to simultaneously estimate the DOA parameter and the unknown parameters of the AR(1) amplitude model. The estimator is compressed into a single-parameter search over the DOA parameter alone. Closed-form expressions of the DA CRB for the DOA parameter alone are derived for fast and slow time-varying channel, respectively. As a special case, the CRB under uncorrelated time-varying channel is also derived. Approximate analytical expressions for the CRB of the DOA alone over low and high SNR are derived. Some properties that highlight how the bound depends on key parameters such as SNR and time-Doppler product are proved. These properties show that the CRB for DOA alone is insensitive to the channel-dependent time-Doppler product for high SNR. Numerical simulation shows that the proposed estimator reaches the CRB for a large range of SNR.

A Proof of Eq. (11)

Under the AR(1) channel amplitude model (3), the $M \times 1$ array observation vector is described by the following state-space equations:

$$\mathbf{y}(n) = h(n)\mathbf{a}(\theta) + \mathbf{n}(n), \tag{27}$$

$$h(n) = \gamma h(n-1) + \sqrt{1-\gamma^2}e(n),$$

where $e(n) \sim \mathcal{N}(0,\sigma_h^2)$. Inserting $h(n)$ in (27), we get

$$\begin{aligned}\mathbf{y}(n) &= \gamma h(n-1)\mathbf{a}(\theta) + \sqrt{1-\gamma^2}\mathbf{a}(\theta)e(n) + \mathbf{n}(n)\\ &= \gamma\mathbf{y}(n-1) + \sqrt{1-\gamma^2}\mathbf{a}(\theta)e(n) + \mathbf{n}(n)\\ &- \gamma\mathbf{n}(n-1),\end{aligned} \tag{28}$$

where we have used $h(n-1)\mathbf{a}(\theta) = \mathbf{y}(n-1) - \mathbf{n}(n-1)$. It is clear from (28) that the dependence of $\mathbf{y}(n)$ on its history $\mathbf{y}^{(n)} = \left(\mathbf{y}(0)^T,\ldots,\mathbf{y}(n-1)^T\right)^T$ for the AR(1) model, is limited to dependence on the previous sample $\mathbf{y}(n-1)$ alone

$$p(\mathbf{y}(n)|\mathbf{y}^{(n)};\boldsymbol{\alpha}) = p(\mathbf{y}(n)|\mathbf{y}(n-1);\boldsymbol{\alpha}), \tag{29}$$

By definition, the normal vectors $\mathbf{n}(n)$ and $\mathbf{e}(n)$ are independent of $\mathbf{y}(n-1)$, and $p(\mathbf{y}(n-1)|\mathbf{y}(n-1);\boldsymbol{\alpha})$ is a constant. Hence, from (28), to derive the conditional distribution $p(\mathbf{y}(n)|\mathbf{y}(n-1);\boldsymbol{\alpha})$, we only need to derive the conditional distribution $p(\mathbf{n}(n-1)|\mathbf{y}(n-1);\boldsymbol{\alpha})$. Let $\mathbf{v}_y$ be a complex vector defined as

$$\mathbf{v}_y \stackrel{\text{def}}{=} \left(\mathbf{y}(n)^T, \mathbf{n}(n)^T\right)^T. \tag{30}$$

Clearly, $\mathbf{v}_y$ is zero-mean complex Gaussian vector with covariance matrix given by

$$\mathbf{C}_v \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{v}_y\mathbf{v}_y^H) = \begin{pmatrix} \mathbf{C}_y & \sigma_n^2\mathbf{I} \\ \sigma_n^2\mathbf{I} & \sigma_n^2\mathbf{I}\end{pmatrix},$$

where $\mathbf{C}_y \stackrel{\text{def}}{=} \sigma_h^2\mathbf{a}(\theta)\mathbf{a}^H(\theta) + \sigma_n^2\mathbf{I}$. Applying the well known Bayes rule equality, the conditional distribution $p(\mathbf{n}(n)|\mathbf{y}(n);\boldsymbol{\alpha})$ can be obtained as

$$p(\mathbf{n}(n)|\mathbf{y}(n);\boldsymbol{\alpha}) = \frac{p(\mathbf{n}(n),\mathbf{y}(n);\boldsymbol{\alpha})}{p(\mathbf{y}(n);\boldsymbol{\alpha})}, \tag{31}$$

where

$$\begin{aligned}p(\mathbf{y}(n);\boldsymbol{\alpha}) &= \frac{1}{\pi^M\det(\mathbf{C}_y)}e^{-\mathbf{y}(n)^H\mathbf{C}_y^{-1}\mathbf{y}(n)}\\ p(\mathbf{n}(n),\mathbf{y}(n);\boldsymbol{\alpha}) &= \frac{1}{\pi^{2M}\det(\mathbf{C}_v)}e^{-\mathbf{v}_y^H\mathbf{C}_v^{-1}\mathbf{v}_y}.\end{aligned}$$

Thus,

$$\begin{aligned}&p(\mathbf{n}(n)|\mathbf{y}(n);\boldsymbol{\alpha}) = \\ &\frac{1}{\pi^M\det(\mathbf{C}_y^{-1})\det(\mathbf{C}_v)}e^{-(\mathbf{v}_y^H\mathbf{C}_v^{-1}\mathbf{v}_y - \mathbf{y}(n)^H\mathbf{C}_y^{-1}\mathbf{y}(n))}.\end{aligned} \tag{32}$$

Using matrix inversion lemma [27], the inverse of the matrix $\mathbf{C}_v$ can be expressed as

$$\mathbf{C}_v^{-1} = \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}(\mathbf{C}_y-\sigma_n^2\mathbf{I})^{-1} & \mathbf{O}\\ \mathbf{O} & \sigma_n^{-2}\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I} & -\mathbf{I}\\ \mathbf{O} & \mathbf{I}\end{pmatrix}. \tag{33}$$

Using (33) and (30), and after some mathematical manipulations, we obtain

$$\begin{aligned}\mathbf{v}_y^H\mathbf{C}_v^{-1}\mathbf{v}_y - \mathbf{y}(n)^H\mathbf{C}_y^{-1}\mathbf{y}(n) &= (\mathbf{n}(n) - \sigma_n^2\mathbf{C}_y^{-1}\mathbf{y}(n))^H\mathbf{B}^{-1}(\mathbf{n}(n)\\ &- \sigma_n^2\mathbf{C}_y^{-1}\mathbf{y}(n))\end{aligned} \tag{34}$$

$$\det(\mathbf{C}_y^{-1})\det(\mathbf{C}_v) = \det(\mathbf{B}), \tag{35}$$

where $\mathbf{B} \stackrel{\text{def}}{=} \sigma_n^2(\mathbf{C}_y - \sigma_n^2\mathbf{I})\mathbf{C}_y^{-1}$. Thus, $\mathbf{n}(n)|\mathbf{y}(n)$ is a complex Gaussian vector with pdf given by

$$p(\mathbf{n}(n)|\mathbf{y}(n);\boldsymbol{\alpha}) = \frac{1}{\pi^M \det(\mathbf{B})} e^{-\mathbf{n}_y(n)^H \mathbf{B}^{-1}\mathbf{n}_y(n)}. \quad (36)$$

where $\mathbf{n}_y(n) \stackrel{\text{def}}{=} \mathbf{n}(n) - \sigma_n^2 \mathbf{C}_y^{-1}\mathbf{y}(n)$. Therefore, the conditional distribution of $\mathbf{y}(n)|\mathbf{y}(n-1)$ is Gaussian and according to (28) its mean and covariance matrix are given by

$$\begin{aligned} \mathrm{E}(\mathbf{y}(n)|\mathbf{y}(n-1)) &= \gamma\mathbf{y}(n-1) \\ &- \gamma\mathrm{E}(\mathbf{n}(n-1)|\mathbf{y}(n-1)) \\ &= \gamma\sigma_h^2\mathbf{C}_y^{-1}\mathbf{a}\mathbf{a}^H\mathbf{y}(n-1) \quad (37) \\ &= \frac{\gamma\sigma_h^2}{M\sigma_h^2+\sigma_n^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1), \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(\mathbf{y}(n)|\mathbf{y}(n-1)) = \mathrm{E}(\mathbf{y}(n)\mathbf{y}(n)^H|\mathbf{y}(n-1)) \\ - \mathrm{E}(\mathbf{y}(n)|\mathbf{y}(n-1)) \quad (38) \\ (\mathrm{E}(\mathbf{y}(n)|\mathbf{y}(n-1)))^H. \end{aligned}$$

To derive the covariance matrix (38), we begin with

$$\begin{aligned} &\mathrm{E}(\mathbf{y}(n)\mathbf{y}(n)^H|\mathbf{y}(n-1)) = \mathrm{E}\left((\gamma\mathbf{y}(n-1)\right. \\ +\ &\left.\sqrt{(1-\gamma^2)}\mathbf{a}(\theta)e(n) + \mathbf{n}(n) - \gamma\mathbf{n}(n-1)\right) \\ &\left(\gamma\mathbf{y}(n-1) + \sqrt{(1-\gamma^2)}\mathbf{a}(\theta)e(n) + \mathbf{n}(n)\right. \\ -\ &\left.\gamma\mathbf{n}(n-1))^H\,|\mathbf{y}(n-1)\right). \quad (39) \end{aligned}$$

Note that $\mathbf{n}(n)$ is independent of $\mathbf{y}(n-1)$, and therefore, also of $\mathbf{n}(n)|\mathbf{y}(n-1)$. We also note that $e(n)$ is independent of $\mathbf{n}(n-1)$, $\mathbf{n}(n)$ and $\mathbf{y}(n-1)$, hence, all the cross terms in (39) except those that involve both $\mathbf{n}(n-1)$ and $\mathbf{y}(n-1)$ vanish.

After some lengthy but straightforward algebraic manipulations, we obtain

$$\begin{aligned} &\mathrm{E}(\mathbf{y}(n)\mathbf{y}(n)^H|\mathbf{y}(n-1)) = \gamma^2\mathbf{y}(n-1)\mathbf{y}(n-1)^H \\ -\ &\gamma^2\mathbf{y}(n-1)\mathrm{E}(\mathbf{n}(n-1)^H|\mathbf{y}(n-1)) \\ -\ &\gamma^2\mathrm{E}(\mathbf{n}(n-1)|\mathbf{y}(n-1))\mathbf{y}(n-1)^H \\ +\ &\gamma^2\mathrm{E}(\mathbf{n}(n-1)\mathbf{n}(n-1)^H|\mathbf{y}(n-1)) \\ +\ &(1-\gamma^2)\mathbf{a}\mathbf{a}^H\mathrm{E}(|e(n)|^2) + \mathrm{E}(\mathbf{n}(n)\mathbf{n}(n)^H) \\ =\ &\gamma^2\mathbf{y}(n-1)\mathbf{y}(n-1)^H - \gamma^2\sigma_n^2\mathbf{y}(n-1)\mathbf{y}(n-1)^H\mathbf{C}_y^{-1} \\ -\ &\gamma^2\sigma_n^2\mathbf{C}_y^{-1}\mathbf{y}(n-1)\mathbf{y}(n-1)^H \\ +\ &\gamma^2\mathbf{B} + \gamma^2\sigma_n^4\mathbf{C}_y^{-1}\mathbf{y}(n-1)\mathbf{y}(n-1)^H\mathbf{C}_y^{-1} \\ +\ &\sigma_h^2(1-\gamma^2)\mathbf{a}\mathbf{a}^H + \sigma_n^2\mathbf{I} \\ =\ &\gamma^2\sigma_h^4(\mathbf{y}(n-1)^H\mathbf{C}_y^{-1}\mathbf{a})\mathbf{C}_y^{-1}\mathbf{a}\mathbf{a}^H\mathbf{y}(n-1)\mathbf{a}^H \\ +\ &\gamma^2\mathbf{B} + \sigma_h^2(1-\gamma^2)\mathbf{a}\mathbf{a}^H + \sigma_n^2\mathbf{I}. \quad (40) \end{aligned}$$

Therefore, using (40) and (37) the conditional covariance matrix can be expressed as

$$\begin{aligned} \mathbf{C} &\stackrel{\text{def}}{=} \mathrm{Cov}(\mathbf{y}(n)|\mathbf{y}(n-1)) \\ &= \mathrm{E}(\mathbf{y}(n)\mathbf{y}(n)^H|\mathbf{y}(n-1)) \\ &- \mathrm{E}(\mathbf{y}(n)|\mathbf{y}(n-1))(\mathrm{E}(\mathbf{y}(n)|\mathbf{y}(n-1)))^H \\ &= \gamma^2\mathbf{B} + \sigma_h^2(1-\gamma^2)\mathbf{a}\mathbf{a}^H + \sigma_n^2\mathbf{I} \\ &= -\gamma^2\sigma_h^4\|\mathbf{a}\|^2\mathbf{C}_y^{-1}\mathbf{a}\mathbf{a}^H + \mathbf{C}_y \\ &= \sigma_h^2\left(1 - \frac{\gamma^2 M\sigma_h^2}{M\sigma_h^2+\sigma_n^2}\right)\mathbf{a}(\theta)\mathbf{a}^H(\theta) + \sigma_n^2\mathbf{I}. \end{aligned}$$

Hence, the pdf of $\mathbf{y}$ can be expressed as

$$p(\mathbf{y};\boldsymbol{\alpha}) = p(\mathbf{y}(0);\boldsymbol{\alpha})\prod_{n=1}^{N-1} p(\mathbf{y}(n)|\mathbf{y}(n-1);\boldsymbol{\alpha}),$$

where

$$p(\mathbf{y}(0);\boldsymbol{\alpha}) = \frac{1}{\pi^M\det(\mathbf{C}_y)} e^{-\mathbf{y}(0)^H\mathbf{C}_y^{-1}\mathbf{y}(0)}$$

$$p(\mathbf{y}(n)|\mathbf{y}(n-1);\boldsymbol{\alpha}) = \frac{1}{\pi^M\det(\mathbf{C})} e^{-\bar{\mathbf{y}}(n)^H\mathbf{C}^{-1}\bar{\mathbf{y}}(n)}$$

and $\bar{\mathbf{y}}(n) \stackrel{\text{def}}{=} \mathbf{y}(n) - \frac{\gamma\sigma_h^2}{M\sigma_h^2+\sigma_n^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1)$. Therefore, the negative log-pdf can be expressed as (after dropping the constant term)

$$\begin{aligned} L(\boldsymbol{\alpha}_n,\theta;\mathbf{y}) = \ln((\det(\mathbf{C}_y))^N(\det(\bar{\mathbf{C}}))^{N-1}) \quad (41) \\ + \mathbf{y}(0)^H\mathbf{C}_y^{-1}\mathbf{y}(0) + \sum_{n=1}^{N-1}\bar{\mathbf{y}}(n)^H\mathbf{C}^{-1}\bar{\mathbf{y}}(n), \end{aligned}$$

where $\bar{\mathbf{C}} \stackrel{\text{def}}{=} \mathbf{I} - \frac{\gamma^2 M\sigma_h^4}{M\sigma_h^2+\sigma_n^2}\mathbf{C}_y^{-1}\mathbf{a}(\theta)\mathbf{a}^H(\theta)$. The matrix $\mathbf{C}_y^{-1}$, $\det(\mathbf{C}_y)$ and $\det(\bar{\mathbf{C}})$ can be written, using matrix determinant and inversion lemmas, as

$$\begin{aligned} \mathbf{C}_y^{-1} &= \frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{M\sigma_n^2(1+\frac{\sigma_n^2}{M\sigma_h^2})}\mathbf{a}\mathbf{a}^H, \quad (42) \\ \det(\mathbf{C}_y) &= \det(\sigma_h^2\mathbf{a}\mathbf{a}^H + \sigma_n^2\mathbf{I}) \\ &= M\sigma_n^{2(M-1)}\sigma_h^2(1+\frac{\sigma_n^2}{M\sigma_h^2}), \\ \det(\bar{\mathbf{C}}) &= 1 - \gamma^2\frac{1}{(1+\frac{\sigma_n^2}{M\sigma_h^2})^2}. \end{aligned}$$

Hence, the first term in (41) can be simplified as

$$\begin{aligned} &\ln((\det(\mathbf{C}_y))^N(\det(\bar{\mathbf{C}}))^{N-1}) = N\ln(M) \\ +\ &N\ln(\sigma_n^{2(M-1)}\sigma_h^2) + N\ln(1+\frac{\sigma_n^2}{M\sigma_h^2}) \\ +\ &(N-1)\ln(1-\gamma^2\frac{1}{(1+\frac{\sigma_n^2}{M\sigma_h^2})^2}). \end{aligned}$$

Consequently, (11) is proved.

B Proof of Result 1

Equating to zero the first derivative of $L'(.)$ in (12) with respect to σ_n^2, σ_h^2 and γ, we get the following high-SNR ML equations

$$\begin{aligned}\frac{\partial L'(\boldsymbol{\alpha};\mathbf{y})}{\partial \sigma_n^2} &= \frac{N(M-1)}{\sigma_n^2} - \frac{1}{\sigma_n^4}\mathbf{y}(0)^H \Pi^{\perp}_{\mathbf{a}(\theta)}\mathbf{y}(0) \\ &- \frac{1}{\sigma_n^4}\sum_{n=1}^{N-1}\tilde{\mathbf{y}}(n)^H \Pi^{\perp}_{\mathbf{a}(\theta)}\tilde{\mathbf{y}}(n) = 0, \quad (43)\end{aligned}$$

$$\begin{aligned}\frac{\partial L'(\boldsymbol{\alpha};\mathbf{y})}{\partial \sigma_h^2} &= \frac{N}{\sigma_h^2} - \frac{1}{M^2\sigma_h^4}\mathbf{y}(0)^H \mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(0) \\ &- \frac{1}{M^2\sigma_h^4(1-\gamma^2)}\sum_{k=1}^{N-1}\tilde{\mathbf{y}}(n)^H\mathbf{a}(\theta). \\ &\mathbf{a}^H(\theta)\tilde{\mathbf{y}}(n) = 0 \quad (44)\end{aligned}$$

$$\begin{aligned}&\frac{\partial L'(\boldsymbol{\alpha};\mathbf{y})}{\partial \gamma} = -\frac{2(N-1)\gamma}{1-\gamma^2} \\ &+ \sum_{k=1}^{N-1}\{-\frac{1}{M}\mathbf{y}(n-1)^H\mathbf{a}(\theta)\mathbf{a}^H(\theta)\tilde{\mathbf{C}}\tilde{\mathbf{y}}(n) \\ &- \frac{1}{M}\tilde{\mathbf{y}}_n^H\tilde{\mathbf{C}}\mathbf{a}(\theta)\mathbf{a}^H(\theta)\mathbf{y}(n-1) \quad (45) \\ &+ \frac{2\gamma}{M(1-\gamma^2)^2\sigma_h^2}\tilde{\mathbf{y}}(n)^H\mathbf{a}(\theta)\mathbf{a}^H(\theta)\tilde{\mathbf{y}}(n)\} = 0.\end{aligned}$$

Solving (43) w.r.t σ_n^2, we obtain the ML estimate of σ_n^2 expressed as function of θ and $\mathbf{y}$ as

$$\hat{\sigma}^2_{n,\mathrm{ML}} = \frac{1}{N(M-1)}\sum_{k=0}^{N-1}\mathbf{y}(n)^H\Pi^{\perp}_{\mathbf{a}(\theta)}\mathbf{y}(n). \quad (46)$$

Next, solving (43) w.r.t σ_h^2, we obtain the following ML estimate of σ_h^2 that depends on θ, γ and $\mathbf{y}$

$$\begin{aligned}\hat{\sigma}^2_{h,\mathrm{ML}} &= \frac{1}{N}\left(k_{3,y}(\theta) + \frac{1}{1-\gamma^2}\left(-\gamma k_{2,y}(\theta)\right.\right. \\ &+ \left.\left.\gamma^2_{ML}k_{1,y}(\theta)\right)\right). \quad (47)\end{aligned}$$

Collecting the coefficients for each order of γ, the ML estimate of γ is given as the solution of the following third-order polynomial:

$$\begin{aligned}&2(N-1)\sigma_h^2\gamma^3 - k_{2,y}(\theta)\gamma^2 + (2k_{1,y}(\theta) \\ -\ &2(N-1)\sigma_h^2)\gamma - k_{2,y}(\theta) = 0, \quad (48)\end{aligned}$$

which depends on θ, σ_h^2 and $\mathbf{y}$. Substituting $\hat{\sigma}^2_{h,\mathrm{ML}}$ given by (47) into (48), we obtain after some easy manipulations the following ML equation in γ

$$\begin{aligned}P(\gamma) &\stackrel{\text{def}}{=} 2k_{4,y}(\theta)\gamma^5 + k_{2,y}(\theta)\gamma^4 - 4k_{4,y}(\theta)\gamma^3 \\ &- 2k_{2,y}(\theta)\gamma^2 + 2k_{4,y}(\theta)\gamma + k_{2,y}(\theta) = 0.\end{aligned}$$

Fortunately, this polynomial can be factored as

$$P(\gamma) = (2k_{4,y}(\theta)\gamma + k_{2,y}(\theta))(\gamma^2-1)^2 = 0.$$

Consequently, the ML estimate of γ is given by

$$\hat{\gamma}_{\mathrm{ML}} = -\frac{k_{2,y}(\theta)}{2k_{4,y}(\theta)}.$$

Substituting this solution into (47), we find the ML estimate of σ_h^2 given by (15).

C Proof of Result 2

To compute the elements of the FIM given by (19), we need the following partial derivatives of the covariance matrix $\mathbf{R}_y$ with respect to the unknown parameters.

$$\begin{aligned}\frac{\partial \mathbf{R}_y}{\partial \theta} &= \mathbf{D}\mathbf{R}_h\mathbf{A}^H + \mathbf{A}\mathbf{R}_h\mathbf{D}^H \quad (49) \\ \frac{\partial \mathbf{R}_y}{\partial \sigma_n^2} &= \mathbf{I} \quad (50) \\ \frac{\partial \mathbf{R}_y}{\partial \sigma_h^2} &= \frac{1}{\sigma_h^2}\mathbf{A}\mathbf{R}_h\mathbf{A}^H \quad (51) \\ \frac{\partial \mathbf{R}_y}{\partial \gamma} &= \mathbf{A}\mathbf{R}_h^{(\gamma)}\mathbf{A}^H, \quad (52)\end{aligned}$$

where $\mathbf{R}_h^{(\gamma)} \stackrel{\text{def}}{=} \frac{\partial \mathbf{R}_h}{\partial \gamma}$, $\mathbf{D} \stackrel{\text{def}}{=} \frac{\partial \mathbf{A}}{\partial \theta} = \mathbf{I}\otimes\mathbf{a}'$, and where $\mathbf{a}' \stackrel{\text{def}}{=} \frac{\partial \mathbf{a}}{\partial \theta}$.

Now, we propose to show that the DOA parameter is decoupled from the other parameters in FIM (i.e., $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_h^2} = 0$, $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_n^2} = 0$ and $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\gamma} = 0$).

Let us define $\tilde{\mathbf{Z}} \stackrel{\text{def}}{=} (M\mathbf{R}_h + \sigma_n^2\mathbf{I})^{-1}$. Applying the well-known matrix inversion lemma (e.g., [27]), we get

$$\mathbf{R}_y^{-1} = \frac{1}{\sigma_n^2}(\mathbf{I} - \mathbf{A}\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{A}^H), \quad (53)$$

and hence, we obtain the following equalities

$$\begin{aligned}\mathbf{R}_y^{-1}\mathbf{A} &= \mathbf{A}\tilde{\mathbf{Z}}, \quad (54) \\ \mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A} &= M\tilde{\mathbf{Z}}. \quad (55)\end{aligned}$$

Using (49) and (50), we obtain

$$\begin{aligned}(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_h^2} &= \mathrm{Tr}\left(\frac{\partial \mathbf{R}_y}{\partial \theta}\mathbf{R}_y^{-1}\frac{\partial \mathbf{R}_y}{\partial \sigma_h^2}\mathbf{R}_y^{-1}\right) \\ &= \frac{1}{\sigma_h^2}\left(\mathrm{Tr}(\mathbf{D}\mathbf{R}_h\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_h\mathbf{A}^H\mathbf{R}_y^{-1})\right. \\ &+ \left.\mathrm{Tr}(\mathbf{R}_h\mathbf{D}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_h\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A})\right) \\ &= \frac{M}{\sigma_h^2}\left(\mathrm{Tr}(\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{A}^H\mathbf{D})\right. \\ &+ \left.\mathrm{Tr}(\tilde{\mathbf{Z}}\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h\mathbf{D}^H\mathbf{A})\right), \quad (56)\end{aligned}$$

where we have used (54) and (55) and $\mathbf{A}^H\mathbf{A} = M\mathbf{I}$. To simplify the calculation, we need the following property that can be proved thanks to the properties of the trace operator (e.g., [27]).

Lemma 1: Let $\mathbf{E}$ and $\mathbf{F}$ be two symmetric matrices, and $\mathbf{G}$ be a diagonal matrix. Then,

$$\begin{aligned}\mathrm{Tr}(\mathbf{EFG}) &= \mathrm{Tr}((\mathbf{FE})^T\mathbf{G}) = \mathrm{Tr}(\mathbf{FEG}^T)\\ &= \mathrm{Tr}(\mathbf{FEG}).\end{aligned}$$

Note that $\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h$ is a symmetric matrix because $\mathbf{R}_h$ and $\tilde{\mathbf{Z}}$ are symmetric matrices. By applying lemma 1 to the first term of (56) (with $\mathbf{E} = \mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h$, $\mathbf{F} = \tilde{\mathbf{Z}}$, and $\mathbf{G} = \mathbf{A}^H\mathbf{D}$), (56) becomes

$$\begin{aligned}(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_h^2} &= \frac{M}{\sigma_h^2}\Big(\mathrm{Tr}\Big(\tilde{\mathbf{Z}}\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h\left(\mathbf{A}^H\mathbf{D}\right.\\ &+ \left.\left.\left.\mathbf{D}^H\mathbf{A}\right)\right)\right) = 0, \qquad (57)\end{aligned}$$

where the last equality results from the key equality

$$\begin{aligned}\frac{d\mathbf{A}^H\mathbf{A}}{d\theta} &= \frac{d(\mathbf{a}^H\mathbf{a})\mathbf{I}}{d\theta} = \mathbf{A}^H\mathbf{D} + \mathbf{D}^H\mathbf{A}\\ &= (\mathbf{a}^H\mathbf{a}' + \mathbf{a}'^H\mathbf{a})\mathbf{I} = 0. \qquad (58)\end{aligned}$$

Following the same steps, the term $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_n^2}$ can be simplified as

$$\begin{aligned}(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\sigma_n^2} &= \mathrm{Tr}\left(\frac{\partial\mathbf{R}_y}{\partial\theta}\mathbf{R}_y^{-1}\frac{\partial\mathbf{R}_y}{\partial\sigma_n^2}\mathbf{R}_y^{-1}\right) \qquad (59)\\ &= \mathrm{Tr}(\mathbf{R}_h\tilde{\mathbf{Z}}^2\mathbf{A}^H\mathbf{D} + \tilde{\mathbf{Z}}^2\mathbf{R}_h\mathbf{D}^H\mathbf{A})\\ &= \mathrm{Tr}(\tilde{\mathbf{Z}}^2\mathbf{R}_h(\mathbf{A}^H\mathbf{D} + \mathbf{D}^H\mathbf{A})) = 0,\end{aligned}$$

where we have used lemma 1 (with $\mathbf{E} = \mathbf{R}_h$, $\mathbf{F} = \tilde{\mathbf{Z}}^2$ and $\mathbf{G} = \mathbf{A}^H\mathbf{D}$) and (58).

After some easy manipulations similar to (57) and (59), the term $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\gamma}$ can be simplified as

$$\begin{aligned}(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\gamma} &= \mathrm{Tr}\left(\frac{\partial\mathbf{R}_y}{\partial\theta}\mathbf{R}_y^{-1}\frac{\partial\mathbf{R}_y}{\partial\gamma}\mathbf{R}_y^{-1}\right) \qquad (60)\\ &= M\,\mathrm{Tr}(\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h^{(\gamma)}\tilde{\mathbf{Z}}\mathbf{A}^H\mathbf{D}\\ &+ \tilde{\mathbf{Z}}\mathbf{R}_h^{(\gamma)}\tilde{\mathbf{Z}}\mathbf{R}_h\mathbf{D}^H\mathbf{A}).\end{aligned}$$

By applying lemma 1 (with $\mathbf{E} = \mathbf{R}_h$, $\mathbf{F} = \tilde{\mathbf{Z}}^2$ and $\mathbf{G} = \mathbf{A}^H\mathbf{D}$) and (58), we get

$$(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\gamma} = \mathrm{Tr}(\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h^{(\gamma)}\tilde{\mathbf{Z}}(\mathbf{A}^H\mathbf{D} + \mathbf{D}^H\mathbf{A})) = 0. \qquad (61)$$

From (57), (59) and (61), the FIM can be written as

$$\mathbf{FIM} = \begin{pmatrix} (\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\theta} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{X} \end{pmatrix},$$

where $\mathbf{X}$ is the FIM block that corresponds to the parameters $(\sigma_h^2, \gamma, \sigma_n^2)^T$.

Hence, we conclude that the DOA parameter is decoupled from the other parameters in FIM, and therefore the CRB for DOA parameter, θ, is given

$$\mathbf{CRB}(\theta) = (\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\theta}^{-1}. \qquad (62)$$

Now, we propose to compute the term $(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\theta}$. After some easy manipulations similar to (57), (59) and (61), we get

$$\begin{aligned}(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\theta} &= \mathrm{Tr}\left(\frac{\partial\mathbf{R}_y}{\partial\theta}\mathbf{R}_y^{-1}\frac{\partial\mathbf{R}_y}{\partial\theta}\mathbf{R}_y^{-1}\right) \qquad (63)\\ &= \mathrm{Tr}\left(\left((\mathbf{a}^H\mathbf{a}')^2\tilde{\mathbf{Z}} + M(\mathbf{D}^H\mathbf{R}_y^{-1}\mathbf{D})\right)(\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h)\right),\end{aligned}$$

where we have used (49), (54) and (55), and the fact that $\mathbf{A}^H\mathbf{D} = (\mathbf{a}^H\mathbf{a}')\mathbf{I}$.

Using (53) and after straightforward algebra manipulations, the first term in (63) can be simplified as

$$2((\mathbf{a}^H\mathbf{d}')^2\tilde{\mathbf{Z}} + M(\mathbf{D}^H\mathbf{R}_z^{-1}\mathbf{D})) = \frac{M\alpha}{\sigma_n^2}.$$

Therefore,

$$(\mathbf{I}_{\boldsymbol{\alpha}})_{\theta,\theta} = \frac{M\alpha}{\sigma_n^2}\mathrm{Tr}(\mathbf{R}_h\tilde{\mathbf{Z}}\mathbf{R}_h).$$

From this last equality and (62), we obtain (20).

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