On the Asymptotic Stabilization of an Anaerobic Digestion Model with Unknown Kinetics*

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Abstract: In this paper we consider a nonlinear model of a biological wastewater treatment process, based on two microbial populations and two substrates. As a result of this process methane is produced. This model is known to be practically validated and reliable. Two feedback control laws are proposed (one of them is adaptive) for asymptotic stabilization of the closed-loop system towards a previously chosen operating point. Computer simulations are reported to compare the effectiveness of the proposed feedbacks.

Key-Words: model of a biological wastewater treatment process, asymptotic stabilization, feedback control, model uncertainty, numerical simulation, comparison results

1 Introduction

Biological wastewater treatment using anaerobic digestion is a process, where microorganisms decompose the organic compounds inside the effluent. The goal is to reduce the pollutant concentration in the outlet stream below a specified value, usually fixed by environmental and safety rules. At the same time this process can also produce valuable energy (methane). The operation of such processes poses a number of practical problems, since anaerobic digestion is known to become easily unstable, see e. g. [2], [13], [16], [17] and the references therein.

In the recent years the dynamic modeling of anaerobic digestion has become an active research area. This is due to the fact that a mathematical model of the plant can be used as a powerful tool to simulate different operating, control and optimization strategies [4], [18], [22]. The design of such models should find a "trade-off" between model complexity and mathematical investigation of the model, especially for control purposes [13].

One of the main drawbacks in the modeling and control of anaerobic digestion lies in the difficulty to monitor on-line the key biological variables of the process and to obtain explicit analytic expressions for the growth rate functions [21]. Thus developing control systems only based on simple measurements and general assumptions on the specific growth rates, that guarantee stability of the process, is of primary importance.

Practical experiments show that feedback control is a very appropriate tool for asymptotic stabilization in the case of model uncertainties. Such a feedback is proposed in [17] for the so-called "single substrate/single biomass" model under general assumptions on the growth rate functions. This approach is further developed in [8].

The present paper considers two approaches for asymptotic stabilization of a four-dimensional nonlinear control system [2], [5], [13], [14], [16], that models a wastewater treatment process. The first approach is based on adaptive feedback stabilization, and the second on a state feedback (non-adaptive) one. In both cases the closed-loop system is globally stabilized towards a previously chosen operating (reference) point. Thereby we don't assume any analytical expressions for the specific growth rates to be given. Moreover, both feedback control laws depend on on-line measurements only.

The paper is organized as follows. Section 2 presents shortly the dynamic model of the wastewater treatment process. The adaptive asymptotic stabilization of the dynamic system towards a previously chosen operating point is studied in Section 3. Section 4 considers the non-adaptive stabilization problem for the same model. In order to prove that the closed-loop system is asymptotically stable, suitable Lyapunov-like functions are constructed explicitly in both cases. Computer simulations are reported in Section 5 to illustrate and to compare the theoretical results of the two approaches. There, the robustness of the two feedback control laws under model uncertainties is demonstrated as well.

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2 Model Description

We consider a model of an anaerobic digestion process, based on two main reactions (cf. e. g. [2], [5], [13], [14], [16]):

(a) acidogenesis, where the organic substrate is degraded into volatile fatty acids (VFA) by acidogenic bacteria;

(b) methanogenesis, where VFA are degraded into methane CH_4 and carbon dioxide CO_2 by methanogenic bacteria.

The mass balance model in a continuously stirred tank bioreactor is described by the following nonlinear system of ordinary differential equations

$$\frac{ds_1}{dt} = u(s_1^i - s_1) - k_1\ \mu_1\ x_1 \tag{1}$$

$$\frac{dx_1}{dt} = (\mu_1 - \alpha u)x_1 \tag{2}$$

$$\frac{ds_2}{dt} = u(s_2^i - s_2) + k_2\ \mu_1 x_1 - k_3\ \mu_2\ x_2 \tag{3}$$

$$\frac{dx_2}{dt} = (\mu_2 - \alpha u)x_2 \tag{4}$$

with output

$$Q = k_4 \mu_2(s_2) x_2. \tag{5}$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 represents the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria respectively. The parameter $\alpha \in (0, 1)$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively (cf. [1], [2], [5], [13], [20]).

We assume that the methane flow rate Q is the measurable output.

The input substrate concentrations s_1^i and s_2^i are assumed to be constant. The dilution rate u is considered as a control input.

The definition of the model parameters is given in Table 1. There the constants m_1, m_2, k_{s_1}, k_{s_2} and k_I are related to the particular expressions of the specific growth rate functions $\mu_1(s_1)$ and $\mu_2(s_2)$, which are used later in Section 5.

The functions μ_1 and μ_2 model the specific growth rates of the microorganisms. We do not assume to know explicit expressions for the latter, we only impose the following general assumptions on μ_1 and μ_2:

Assumption A1. $\mu_j(s_j)$ is defined for $s_j \in [0, +\infty)$, $\mu_j(0) = 0$, $\mu_j(s_j) > 0$ for $s_j > 0$; $\mu_j(s_j)$ is continuously differentiable and bounded for all $s_j \in [0, +\infty)$, $j = 1, 2$.

Table 1: Definition of the model parameters

s_1	concentration of chemical oxygen demand (COD) [g/l]
s_2	concentration of volatile fatty acids (VFA) [mmol/l]
x_1	concentration of acidogenic bacteria [g/l]
x_2	concentration of methanogenic bacteria [g/l]
u	dilution rate [day^{-1}]
s_1^i	influent concentration s_1 [g/l]
s_2^i	influent concentration s_2 [mmol/l]
k_1	yield coefficient for COD degradation [g COD/(g x_1)]
k_2	yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	yield coefficient for VFA consumption [mmol VFA/(g x_2)]
k_4	coefficient [l^2/g]
m_1	maximum acidogenic biomass growth rate [day^{-1}]
m_2	maximum methanogenic biomass growth rate [day^{-1}]
k_{s_1}	saturation parameter associated with s_1 [g COD/l]
k_{s_2}	saturation parameter associated with s_2 [mmol VFA/l]
k_I	inhibition constant associated with s_2 [(mmol VFA/l)$^{1/2}$]
α	proportion of dilution rate reflecting process heterogeneity
Q	methane gas flow rate

3 Adaptive Asymptotic Stabilization

In this section we shall construct an adaptive stabilizing controller of (1)–(4). First we make the following assumption:

Assumption A2. The biological oxygen demand (BOD) $\frac{k_2}{k_1}s_1 + s_2$ is on-line measurable.

For the practical application it is worth to note that BOD is online measurable. This fact is discussed in details in [5]. The interested reader can find an overview of existing observers in [1], [2]. Information about more specialized biosensors is given in [6] and the references therein.

Let us fix an operating (reference) point $\bar{s}$,

$$\bar{s} \in (0, s^i) \quad \text{with} \quad s^i := \frac{k_2}{k_1} s_1^i + s_2^i. \tag{6}$$

Assumption A3: There exists a point $\bar{s}_1$ such that

$$\mu_1(\bar{s}_1) = \mu_2\left(\bar{s} - \frac{k_2}{k_1}\bar{s}_1\right) > 0, \quad \bar{s}_1 \in \left(0, s_1^i\right).$$

Assumption A3 is called regulability [13] of the system: it means that there exists (at least one) nontrivial equilibrium of the system (1)–(4), corresponding to a constant value of the dilution rate $\bar{u} > 0$. Define further

$$\begin{aligned} \bar{s}_2 &= \bar{s} - \frac{k_2}{k_1}\bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \\ \bar{x}_2 &= \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1}{\alpha k_3} = \frac{s^i - \bar{s}}{\alpha k_3}. \end{aligned} \tag{7}$$

It is straightforward to see that the point

$$\bar{\zeta} := (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is an equilibrium point for the system (1)–(4). Our goal is to construct an adaptive feedback law to asymptotically stabilize the system (1)–(4) to $\bar{\zeta}$.

Denoting $s := \frac{k_2}{k_1}s_1 + s_2$, we define the sets

$$\begin{aligned} \Omega_0 &= \{(s_1, x_1, s_2, x_2) | \\ &\qquad s_1 > 0,\ x_1 > 0,\ s_2 > 0,\ x_2 > 0\}, \\ \Omega_1 &= \{(s_1, x_1, s_2, x_2) | \\ &\qquad s_1 + k_1 x_1 \le \frac{s_1^i}{\alpha},\ s + k_3 x_2 \le \frac{s^i}{\alpha}\right\}, \\ \Omega_2 &= \left\{\left(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2\right) | \right. \\ &\qquad \left. 0 < s_1 < \frac{k_1}{k_2}\bar{s},\ x_1 > 0\right\}, \\ \Omega &= \Omega_0 \cap \Omega_1. \end{aligned}$$

Assumption A4: Let the inequality $\mu_1'(s_1) + \frac{k_2}{k_1} \cdot \mu_2'\left(\bar{s} - \frac{k_2}{k_1}s_1\right) > 0$ be satisfied on the set $\Omega \cap \Omega_2$.

Assumption A4 is technical and is used in the proofs of Theorem 1 and Theorem 2 below. It will be discussed in more details later in Section 5, where the growth rates μ_1 and μ_2 are specified as the Monod and Haldane law and numerical values for the model coefficient are introduced.

Denote

$$\bar{\beta} = \frac{1}{\alpha k_4 \bar{x}_2} = \frac{k_3}{k_4(s^i - \bar{s})} \tag{8}$$

and let $\beta^- > 0$ and $\beta^+ > 0$ be arbitrary real numbers such that $\bar{\beta} \in (\beta^-, \beta^+)$.

Following [2], [17], we extend the system (1)–(4) by adding the differential equation

$$\frac{d\beta}{dt} = -C(\beta - \beta^-)(\beta^+ - \beta) k_4 \, \mu_2(s_2) \, x_2 \, (s - \bar{s}), \tag{9}$$

where $C > 0$ is a constant.

We consider the control system (1)–(4) and (9) in the augmented state space (ζ, β) with $\zeta = (s_1, x_1, s_2, x_2)$ and define the following continuous feedback control law

$$\kappa_1(\zeta, \beta) := k_4 \beta \, \mu_2(s_2) \, x_2. \tag{10}$$

Then the following theorem holds true (a similar assertion can be found in [9] and [10]):

Theorem 1. Let us fix an arbitrary reference point $\bar{s} \in (0, s^i)$. Let Assumptions A1, A2, A3 and A4 be satisfied. Then the feedback (10) stabilizes asymptotically the control system (1)–(4), (9) to the point $(\bar{\zeta}, \bar{\beta})$ for each starting point $\zeta_0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ with

$$s_0 > (s^i - \bar{s}) \cdot \ln\left(\frac{s^i}{s^i - s_0}\right), \tag{11}$$

where $s_0 = s_2^0 + \frac{k_2}{k_1}s_1^0$ and $\beta_0 \in (\beta^-, \beta^+)$.

Proof. Let us fix an arbitrary point $\zeta_0 \in \Omega_0$ and a positive value $u_0 > 0$ for the control. According to Lemma 1 from [13] there exists time $T > 0$ such that the value of the corresponding trajectory of (1)–(4) for $t = T$ belongs to the set Ω, i. e. the trajectory of (1)–(4) starting from the point ζ_0 enters the set Ω after a finite time. For that reason we are studying the control system (1)–(4) after this moment of time, i. e. we assume that the starting point belongs to the set Ω.

Let Σ denote the closed-loop system obtained from (1)–(4) and (9) by substituting the control variable u by the feedback $\kappa_1(\zeta, \beta)$ and let

$$\widetilde{\Omega} := \overline{\Omega} \times (\beta^-, \beta^+),$$

where $\overline{\Omega}$ is the closure of the set Ω. Then one can directly check that $\widetilde{\Omega}$ is positively invariant (cf. [7]) with respect to the trajectories of Σ.

Define the following function

$$V(\zeta, \beta) = (s - \bar{s} + k_3(x_2 - \bar{x}_2))^2 + \Gamma \int_{\bar{s}}^{s} \frac{v - \bar{s}}{s^i - v} dv + \frac{1}{C} \int_{\bar{\beta}}^{\beta} \frac{w - \bar{\beta}}{(w - \beta^-)(\beta^+ - w)} dw,$$

where the parameter $\Gamma > 0$ will be determined later. Clearly, the values of this function are nonnegative.

In what follows we shall subdivide the proof of the theorem in four steps for better clarity and readability.

Step 1. There exists a number $\delta > 0$ such that $V(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0) < V(s_1, x_1, s_2, x_2, \beta)$ for each point $(s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega}$ with $0 \le \frac{k_2}{k_1}s_1 + s_2 \le \delta$.

It is straightforward to see that

$$\int_{\bar{s}}^{s} \frac{v-\bar{s}}{s^i-v}dv = \bar{s} - s - (s^i - \bar{s}) \cdot \ln\left(\frac{s^i - s}{s^i - \bar{s}}\right).$$

If s_0 satisfies the inequality (11), then there exists $\varepsilon > 0$ such that

$$\int_{\bar{s}}^{s_0} \frac{v-\bar{s}}{s^i-v}dv + \varepsilon \le \int_{\bar{s}}^{0} \frac{v-\bar{s}}{s^i-v}dv. \tag{12}$$

The boundedness of the set $\widetilde{\Omega}$ implies the existence of a sufficiently large constant $\Gamma > 0$ such that

$$\begin{aligned}&(s_0-\bar{s}+k_3(x_2^0-\bar{x}_2))^2+\frac{1}{C}\int_{\bar{\beta}}^{\beta_0}\frac{(w-\bar{\beta})\,dw}{(w-\beta^-)(\beta^+-w)}\\ &< (-\bar{s}+k_3(x_2-\bar{x}_2))^2+\\ &\qquad \frac{1}{C}\int_{\bar{\beta}}^{\beta}\frac{(w-\bar{\beta})\,dw}{(w-\beta^-)(\beta^+-w)}+\Gamma\varepsilon\end{aligned}$$

for each point $(0, x_1, 0, x_2, \beta) \in \widetilde{\Omega}$. Multiplying both sides of (12) by Γ and adding to the last inequality, we obtain

$$V(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0) < V(0, x_1, 0, x_2, \beta).$$

The continuity of the function V implies the existence of $\delta > 0$ such that

$$V(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0) < V(s_1, x_1, s_2, x_2, \beta) \tag{13}$$

for each point $(s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega}$ with $0 \le \frac{k_2}{k_1}s_1 + s_2 \le \delta$; this proves Step 1.

Denote by $\dot{V}(\zeta, \beta)$ the Lie derivative of the function V with respect to the right-hand side of the closed-loop system Σ at the point (ζ, β). Then it is easy to see that for each point $(\zeta, \beta) \in \widetilde{\Omega}$ with $\zeta = (s_1, x_1, s_2, x_2)$ the following presentation holds true:

$$\begin{aligned}&\dot{V}(\zeta,\beta) =\\ &\quad -\kappa_1(\zeta,\beta)\left(2+\Gamma\,\frac{k_3}{k_4\beta(s^i-s)(s^i-\bar{s}))}\right)(s-\bar{s})^2\\ &\quad -2(1+\alpha)k_3\cdot\kappa_1(\zeta,\beta)(s-\bar{s})(x_2-\bar{x}_2)\\ &\quad -2\alpha k_3^2\cdot\kappa_1(\zeta,\beta)(x_2-\bar{x}_2)^2.\end{aligned}$$

The boundedness of the set $\widetilde{\Omega}$ implies the existence of a sufficiently large constant $\Gamma > 0$ so that

$$\dot{V}(\zeta, \beta) \le 0 \ \text{ for each point } \ (\zeta, \beta) \in \widetilde{\Omega}. \tag{14}$$

In what follows we shall use a refinement of the LaSalle invariance principle (cf. e. g. [15]) recently obtained in [3] (cf. also [12] and [19], where similar stabilizability problems are studied). First we shall recall some notions. Let us denote by $\phi(t, \zeta, \beta)$ the value of the trajectory of the closed-loop system Σ calculated at time t starting from the point $(\zeta, \beta) \in \widetilde{\Omega}$. The positive limit set (or ω-limit set) of the solution $\phi(t, \zeta, \beta)$ of the closed-loop system Σ is defined as

$$\begin{aligned}&L^+(\zeta,\beta) = \Big\{(\tilde{\zeta},\tilde{\beta}):\ \text{there exists a sequence}\\ &\{t_n\}\to+\infty \text{ with } (\tilde{\zeta},\tilde{\beta}) = \lim_{t_n\to+\infty}\phi(t_n,\zeta,\beta)\Big\}.\end{aligned}$$

Step 2. The ω-limit set $L^+(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0)$ belongs to the set

$$\widetilde{\Omega}_2 := \left\{(s_1, x_1, s_2, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}:\ \frac{k_2}{k_1}s_1 + s_2 = \bar{s}\right\}.$$

According to the extension of the LaSalle invariance principle [3], every solution of Σ starting from a point of $\widetilde{\Omega}$ is defined on the interval $[0, +\infty)$ and approaches the largest invariant set (with respect to Σ) which is contained in one connected component of the closure of the set, where the Lie derivative $\dot{V}$ of V with respect to the right-hand side of Σ is equal to zero.

Let $(\tilde{s}_1, \tilde{x}_1, \tilde{s}_2, \tilde{x}_2, \tilde{\beta})$ be an arbitrary point from the set $L^+(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0)$. We set $\tilde{s} := \frac{k_2}{k_1}\tilde{s}_1 + \tilde{s}_2$ and $\tilde{\zeta} := (\tilde{s}_1, \tilde{x}_1, \tilde{s}_2, \tilde{x}_2)$.

If we assume that $\kappa_1(\tilde{\zeta}, \tilde{\beta}) = 0$, then the inequalities (13) and (14) imply that $\tilde{s} > \delta > 0$, and hence $\tilde{x}_2 = 0$. But then there exists time $T > 0$ with $x_2(t) < \frac{1}{\alpha k_4 \beta^-}$ for each $t \ge T$. For these values of t the differential equation (4) for x_2 becomes

$$\frac{d}{dt}x_2(t) = \mu_2(s_2(t))(1 - \alpha k_4 \beta(t))x_2(t) > 0.$$

So, $0 = \tilde{x}_2$ can not be a limit of $x_2(t)$ as $t \to \infty$. The obtained contradiction shows that $\kappa_1(\tilde{\zeta}, \tilde{\beta}) \ne 0$. This and the equality $\dot{V}(\tilde{\zeta}, \tilde{\beta}) = 0$ imply $\tilde{x}_2 = \bar{x}_2$, $\tilde{s} = \bar{s}$ and $\tilde{\beta} = \bar{\beta}$. Hence $L^+(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0)$ is a subset of the set $\widetilde{\Omega}_2$ which is equivalent to

$$\widetilde{\Omega}_2 = \left\{\left(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}\right) \in \widetilde{\Omega}\right\}$$

and this proves Step 2.

Step 3. The ω-limit set $L^+(s_1^0, x_1^0, s_2^0, x_2^0, \beta^0)$ coincides with the equilibrium point $(\bar{\zeta}, \bar{\beta})$ of Σ.

One can directly check that the set $\widetilde{\Omega}_2$ is invariant with respect to the trajectories of Σ. Using (7), (8) and (10), the dynamics of Σ on the set $\widetilde{\Omega}_2$ can be described by the following system

$$\begin{aligned} \frac{ds_1}{dt} &= \frac{1}{\alpha}\,\chi(s_1)(s_1^i - s_1) - k_1\mu_1(s_1)x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi(s_1))x_1, \end{aligned} \tag{15}$$

where $\chi(s_1) := \mu_2\left(\bar{s} - \dfrac{k_2}{k_1}s_1\right)$. Taking into account that $\bar{s} = \dfrac{k_2}{k_1}\bar{s}_1 + \bar{s}_2$ and $s_1^i = \bar{s}_1 + \alpha k_1 \bar{x}_1$, (15) can be rewritten as follows:

$$\begin{aligned} \frac{ds_1}{dt} &= -\frac{1}{\alpha}\,\chi(s_1)\cdot(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1)) \\ &\quad - k_1\left(\mu_1(s_1) - \chi(s_1)\right)\cdot x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi(s_1))\cdot x_1. \end{aligned}$$

Consider the function

$$\begin{aligned} W(\zeta, \beta) &= (s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 \\ &\quad + \alpha(1-\alpha)k_1^2(x_1 - \bar{x}_1)^2. \end{aligned}$$

Clearly, this function depends only on the variables s_1 and x_1 and takes only nonnegative values; moreover, for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta})$ from the set $\widetilde{\Omega}_2$, the following equality holds true:

$$\begin{aligned} &\dot{W}(s_1, x_1) = \\ &-\frac{2}{\alpha}\chi(s_1)(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 \\ &- 2(1-\alpha)k_1 x_1(s_1 - \bar{s}_1)(\mu_1(s_1) - \chi(s_1)). \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &\mu_1(s_1) - \chi(s_1) = \mu_1(s_1) - \mu_2\left(\bar{s} - \tfrac{k_2}{k_1}s_1\right) \\ &= \mu_1(s_1) - \mu_2\left(\bar{s}_2 - (s_1 - \bar{s}_1)\tfrac{k_2}{k_1}\right) \\ &= \mu_1(\bar{s}_1) + \int_{\bar{s}_1}^{s_1}\mu_1'(\theta)\,d\theta \\ &-\mu_2(\bar{s}_2) + \frac{k_2}{k_1}\int_{\bar{s}_1}^{s_1}\mu_2'\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\,d\theta \\ &= \int_{\bar{s}_1}^{s_1}\left(\mu_1'(\theta) + \frac{k_2}{k_1}\mu_2'\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\right)\,d\theta, \end{aligned}$$

and by means of Assumption A3 it follows that

$$(s_1 - \bar{s}_1)\int_{\bar{s}_1}^{s_1}\left(\mu_1'(\theta) + \frac{k_2}{k_1}\mu_2'\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\right)d\theta \ge 0.$$

From this inequality and from (16) we obtain that

$$\dot{W}(s_1, x_1) \le 0 \tag{17}$$

for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}_2$.

Let us fix an arbitrary point (ζ_0, β_0) from $\widetilde{\Omega}$ with $\zeta_0 = (s_1^0, x_1^0, s_2^0, x_2^0)$. The invariance of the bounded set $\widetilde{\Omega}$ with respect to the trajectories of Σ implies that the ω-limit set $L^+(\zeta_0, \beta_0)$ is a nonempty compact connected invariant set. Moreover, it follows from the LaSalle invariance principle that $L^+(\zeta_0, \beta_0)$ is a subset of $\widetilde{\Omega}_2$. Using (17) it is possible to obtain a better estimate of the location of $L^+(\zeta_0, \beta_0)$. Namely, Theorem 6 from [3] implies that $L^+(\zeta_0, \beta_0)$ is contained in one connected component of the set

$$\begin{aligned} L^\infty := \Big\{&(s_1, x_1, \bar{s} - \tfrac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}_2 : \\ &\dot{W}(s_1, x_1, \bar{s} - \tfrac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) = 0\Big\}. \end{aligned}$$

Let $(\tilde{\zeta}, \bar{\beta})$ be an arbitrary point from the set $L^+(\zeta_0, \beta_0)$, where $\tilde{\zeta} = (\tilde{s}_1, \tilde{x}_1, \tilde{s}_2, \tilde{x}_2)$. Then the following two cases are possible: (i) $\tilde{s}_1 = 0$ and $\tilde{x}_1 = 0$; (ii) $\tilde{s}_1 = \bar{s}_1$ and $\tilde{x}_1 = \bar{x}_1$.

Let us assume that case (i) holds true. Then $\lim\limits_{t\to\infty}\chi(s_1(t)) = 0$, and Assumption A1 implies that $\lim\limits_{t\to\infty}s_1(t) = \dfrac{k_1}{k_2}\bar{s}$. So, there exist $\gamma > 0$ and $T > 0$ such that $\mu_1(s_1(t)) - \chi(s_1(t)) \ge \gamma$ for each $t \ge T$. For these values of t the differential equation for the state variable x_1 in (15) implies that

$$\frac{d}{dt}x_1(t) = (\mu_1(s_1(t)) - \chi(s_1(t)))x_1(t) \ge \gamma x_1(t).$$

Hence $x_1(t) \ge e^{\gamma(t-T)}x_1(T) > 0$ for each $t \in [T, \infty)$ and $x_1(t)$ can not tend to zero as $t \to \infty$. The obtained contradiction leads to the existence of a constant $\chi^{min} > 0$ such that $\chi(s_1(t)) \ge \chi^{min}$ for each $t \ge 0$. Then Lemma 1 from [13] implies the existence of $x_1^{min} > 0$ and $s_1^{min} > 0$ such that $x_1(t) \ge x_1^{min} > 0$ and $s_1(t) \ge s_1^{min} > 0$ for each $t > 0$. So we obtain that case (i) is impossible. Hence the case (ii) is fulfilled. From here it follows that $L^\infty = \{(\bar{\zeta}, \bar{\beta})\}$, and therefore $L^+(\zeta_0, \beta_0) = \{(\bar{\zeta}, \bar{\beta})\}$.

Step 4. The point $(\bar{\zeta}, \bar{\beta})$ is a Lyapunov stable equilibrium of Σ.

Let $\varepsilon > 0$ be arbitrary real number and $(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta})$ be an arbitrary point from the set $\widetilde{\Omega}_2$. Assume that $W(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) = \varepsilon$. Then the inequalities (14) and (17), the continuity of the function W and of the right-hand side of the closed-loop system Σ imply the existence of

$\tilde{\varepsilon}$ with $0 < \tilde{\varepsilon} < \varepsilon$, such that the Lie derivative $\dot{W}(s_1, x_1, s_2, x_2, \beta)$ of the function W with respect to Σ at the point $(s_1, x_1, s_2, x_2, \beta)$ is negative whenever $V(s_1, x_1, s_2, x_2, \beta) \le \tilde{\varepsilon}$. Let us choose an arbitrary sufficiently small positive real number $\varepsilon > 0$, determine as above the number $\tilde{\varepsilon}$ with $0 < \tilde{\varepsilon} < \varepsilon$ and define the set

$$K(\varepsilon) = \Big\{(s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega} : W(s_1, x_1, \bar{s} - \tfrac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) \le \varepsilon\Big\} \bigcap \Big\{(s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega} : V(s_1, x_1, s_2, x_2, \beta) \le \tilde{\varepsilon}\Big\}.$$

Using the definitions of $V(\zeta, \beta)$ and $W(s_1, x_1)$ one can directly check that there exists a constant $\varrho > 0$ such that the ball $B_{\varrho\sqrt{\varepsilon}}(\bar{\zeta}, \bar{\beta})$ centered at the point $(\bar{\zeta}, \bar{\beta})$ with radius $\varrho\sqrt{\varepsilon}$ contains the set $K(\varepsilon)$. Moreover, inequalities (14) and (17) imply that the set $K(\varepsilon)$ is invariant with respect to the trajectories of Σ. This shows that the point $(\bar{\zeta}, \bar{\beta})$ is a Lyapunov stable equilibrium point for the closed-loop system Σ and completes the proof of the theorem. □

4 Asymptotic Stabilization Based on State Feedback Control Law

Consider the control system (1)–(4) in the state space $\zeta = (s_1, x_1, s_2, x_2)$.

Define $s^i := \dfrac{k_2}{k_1} s_1^i + s_2^i$ and let the following assumption be satisfied:

Assumption A5. Lower bounds s^{i-} and k_4^- for the values of s^i and k_4, as well as an upper bound k_3^+ for the value of k_3 are known.

Define the following feedback control law:

$$\kappa_2(\zeta) = \beta\, k_4\, \mu_2(s_2)\, x_2 \quad \text{with } \beta \in \left(\frac{k_3^+}{s^{i-} \cdot k_4^-},\ +\infty\right). \tag{18}$$

Denote again by Σ the closed-loop system obtained from (1)–(4) by substituting the control variable u by the feedback $\kappa_2(\zeta)$ from (18).

Choose some $\beta \in \left(\dfrac{k_3^+}{s^{i-} \cdot k_4^-},\ +\infty\right)$ and let $\bar{s} := s^i - \dfrac{k_3}{\beta k_4}$; obviously, $\bar{s}$ belongs to the interval $(0, s^i)$.

Find $\bar{s}_1$ according to Assumption A3 and define

$$\bar{s}_2 = \bar{s} - \frac{k_2}{k_1}\bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad \bar{x}_2 = \frac{1}{\alpha\beta k_4}. \tag{19}$$

It is straightforward to see that the point

$$\bar{p} := (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is an equilibrium point for Σ. We shall prove below that the feedback law (18) asymptotically stabilizes the closed-loop system to $\bar{p}$ (cf. [11], where the same assertion and an extremum seeking algorithm are presented).

Theorem 2. Let Assumptions A1, A3, A4 and A5 be satisfied. Let us fix an arbitrary number $\beta \in \left(\dfrac{k_3^+}{s^{i-} \cdot k_4^-}, +\infty\right)$ and let $\bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ be the corresponding equilibrium point. Then the feedback control law $\kappa_2(\cdot)$ defined by (18) stabilizes asymptotically the control system (1)–(4) to the point $\bar{p}$ for each starting point ζ_0 from Ω_0.

The **Proof** follows the main steps in the proof of Theorem 1.

Similar arguments as in the beginning of the proof of Theorem 1 show that we can consider the control system (1)–(4) only in the set Ω.

Denote $s := \dfrac{k_2}{k_1} s_1 + s_2$. Then it is not difficult to see that the following ordinary differential equations are satisfied:

$$\begin{aligned} \frac{ds}{dt} &= -\beta k_4 \mu_2(s_2) x_2 (s - \bar{s}) \\ \frac{dx_2}{dt} &= -\alpha\beta k_4 \mu_2(s_2) x_2 (x_2 - \bar{x}_2). \end{aligned} \tag{20}$$

Integrating the equations (20), one obtains

$$s(t) = \bar{s} + (s(0) - \bar{s}) e^{-\int_0^t \beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau}$$

and

$$x_2(t) = \bar{x}_2 + (x_2(0) - \bar{x}_2) e^{-\int_0^t \alpha\beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau}$$

Since the integrands are strictly positive, we have that for each $t > 0$ the following inequalities hold true

$$\max\{s(0), \bar{s}\} \ge s(t) \ge \min\{s(0), \bar{s}\} > 0 \tag{21}$$

and

$$\max\{x_2(0), \bar{x}_2\} \ge x_2(t) \ge \min\{x_2(0), \bar{x}_2\} > 0. \tag{22}$$

Define the function

$$V(\zeta) = (s - \bar{s})^2 + (x_2 - \bar{x}_2)^2.$$

Clearly, the values of this function are nonnegative. If we denote by $\dot{V}(\zeta)$ the Lie derivative of V with respect to the right-hand side of (20) then for each point ζ of Ω,

$$\begin{aligned}\dot V(\zeta) &= -\beta k_4\mu_2(s_2)x_2(s-\bar s)^2\\ &\quad -\alpha\beta k_4\mu_2(s_2)x_2(x_2-\bar x_2)^2 \le 0.\end{aligned}$$

Applying LaSalle's invariance principle (cf. [15]), it follows that every solution of Σ starting from a point of Ω is defined in the interval $[0,+\infty)$ and approaches the largest invariant set with respect to Σ, which is contained in the set Ω_∞, where Ω_∞ is the closure of the set $\Omega\cap\Omega_2$. Taking into account the estimates (21) and (22), the dynamics of Σ on Ω_∞ can be described in the following way

$$\begin{aligned}\frac{ds_1}{dt} &= \frac{1}{\alpha}\,\chi(s_1)(s_1^i-s_1)-k_1\mu_1(s_1)x_1\\ \frac{dx_1}{dt} &= (\mu_1(s_1)-\chi(s_1))x_1,\end{aligned}\tag{23}$$

where $\chi(s_1) := \mu_2\left(\bar s-\dfrac{k_2}{k_1}s_1\right)$ (remind that $\bar s := s^i-\dfrac{k_3}{\beta k_4}$). According to (19) we have that $\bar s = \dfrac{k_2}{k_1}\bar s_1+\bar s_2$ and $s_1^i=\bar s_1+\alpha k_1\bar x_1$. Then (23) can be written as follows:

$$\begin{aligned}\frac{ds_1}{dt} &= -\frac{1}{\alpha}\,\chi(s_1)\cdot(s_1-\bar s_1+\alpha k_1(x_1-\bar x_1))\\ &\quad -k_1\left(\mu_1(s_1)-\chi(s_1)\right)\cdot x_1\\ \frac{dx_1}{dt} &= (\mu_1(s_1)-\chi(s_1))\cdot x_1.\end{aligned}$$

Consider the function

$$\begin{aligned}W(s_1,x_1) &= (s_1-\bar s_1+\alpha k_1(x_1-\bar x_1))^2\\ &\quad +\alpha(1-\alpha)k_1^2(x_1-\bar x_1)^2.\end{aligned}\tag{24}$$

This function takes nonnegative values. It is straightforward to see that for each point $\left(s_1,x_1,\bar s-\frac{k_2}{k_1}s_1,\bar x_2\right)\in\Omega_\infty$,

$$\begin{aligned}\dot W(s_1,x_1) &= -\frac{2}{\alpha}\chi(s_1)(s_1-\bar s_1+\alpha k_1(x_1-\bar x_1))^2\\ &\quad -2(1-\alpha)k_1x_1(s_1-\bar s_1)(\mu_1(s_1)-\chi(s_1)).\end{aligned}$$

Assumptions A3 and A4 imply

$$\begin{aligned}\mu_1(s_1)-\chi(s_1) &= \mu_1(s_1)-\mu_2\left(\bar s-\frac{k_2}{k_1}s_1\right)\\ &= \int_{\bar s_1}^{s_1}\left(\mu_1'(\theta)+\frac{k_2}{k_1}\mu_2'\left(\bar s_2-(\theta-\bar s_1)\frac{k_2}{k_1}\right)\right)d\theta,\end{aligned}$$

and therefore

$$\dot W(s_1,x_1)\le 0$$

for each point $\left(s_1,x_1,\bar s-\dfrac{k_2}{k_1}s_1,\bar x_2\right)$ from Ω_∞.

To complete the proof we can use the same arguments as in Step 4 of Theorem 1. □

5 Numerical Simulation and Comparison Results

In the computer simulation, we consider for $\mu_1(s_1)$ and $\mu_2(s_2)$ the Monod and the Haldane model functions for the specific growth rates, which are used in the original model [1], [5], [13], [14]:

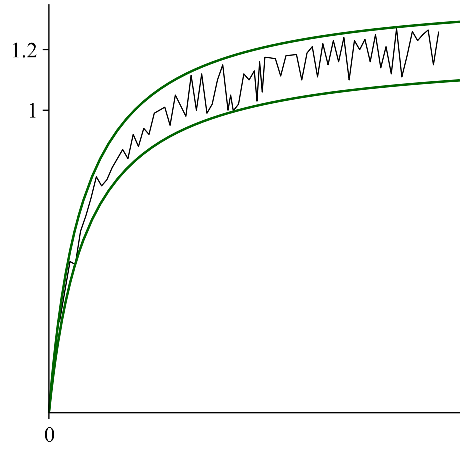
$$\begin{aligned}\mu_1(s_1) &= \frac{m_1s_1}{k_{s_1}+s_1},\\ \mu_2(s_2) &= \frac{m_2s_2}{k_{s_2}+s_2+\left(\dfrac{s_2}{k_I}\right)^2}.\end{aligned}\tag{25}$$

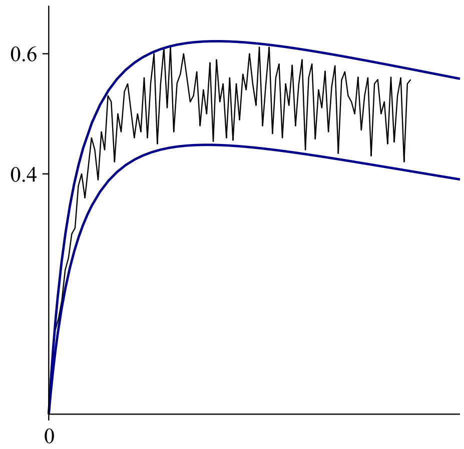
Obviously, $\mu_1(s_1)$ and $\mu_2(s_2)$ satisfy Assumption A1: $\mu_1(s_1)$ and $\mu_2(s_2)$ are continuously differentiable and bounded: $\mu_1(s_1)$ is monotone increasing and $\mu_1(s_1)<m_1$ for all $s_1\ge 0$; $\mu_2(s_2)$ takes its maximum at the point $s_2^m=k_I\sqrt{k_{s_2}}$.

Simple derivative calculations show that if $\bar s$ is chosen such that $0<\bar s\le s_2^m$ then $\mu_2'\left(\bar s-\frac{k_2}{k_1}s_1\right)\ge 0$ holds true thus Assumption A4 is satisfied. Moreover, if the point $\bar s$ is sufficiently small, then Assumptions A3 and A4 are simultaneously satisfied.

For practical applications, the condition (11) in Theorem 1 can be simplified. Remember that $s_0 = s_2^0+\dfrac{k_2}{k_1}s_1^0$ and $(s_1^0,x_1^0,s_2^0,x_2^0)$ is a starting point from Ω_0. Let us choose $s_0 := \gamma\, s^i$ with $\gamma\in(0,1)$. Straightforward calculations deliver that (11) is equivalent with $\gamma > \left(1-\dfrac{\bar s}{s^i}\right)\ln\left(1+\dfrac{\gamma}{1-\gamma}\right)$; moreover, if we choose γ such that $\gamma>\left(1-\dfrac{s_2^m}{s^i}\right)\ln\left(1+\dfrac{\gamma}{1-\gamma}\right)$, then (11) will also be fulfilled for any $0<\bar s\le s_2^m$.

Usually the formulation of the specific growth rates is based on experimental results, and therefore it is not possible to have an exact analytic form of these functions, but only some quantitative bounds, see the figures below.





Assume that we know bounds for $\mu_1(s_1)$ and $\mu_2(s_2)$, i. e.

$\mu_j(s_j) \in [\mu_j(s_j)] = [\mu_j^-(s_j), \mu_j^+(s_j)]$ for all $s_j \geq 0$, $j = 1, 2$.

This uncertainty can be simulated by assuming in (25) that instead of exact values for the kinetic coefficients m_1, k_{s_1}, m_2, k_{s_2} and k_I we have compact intervals for them:

$$m_1 \in [m_1] = [m_1^-, m_1^+], \qquad k_{s_1} \in [k_{s_1}] = [k_{s_1}^-, k_{s_1}^+]$$

$$m_2 \in [m_2] = [m_2^-, m_2^+], \qquad k_{s_2} \in [k_{s_2}] = [k_{s_2}^-, k_{s_2}^+],$$

$$k_I \in [k_I] = [k_I^-, k_I^+].$$

Then

$$\begin{aligned}\mu_1^-(s_1) &= \frac{m_1^- s_1}{k_{s_1}^+ + s_1}, \quad \mu_1^+(s_1) = \frac{m_1^+ s_1}{k_{s_1}^- + s_1},\\ \mu_2^-(s_2) &= \frac{m_2^- s_2}{k_{s_2}^+ + s_2 + \left(\frac{s_2}{k_I^-}\right)^2},\\ \mu_2^+(s_2) &= \frac{m_2^+ s_2}{k_{s_2}^- + s_2 + \left(\frac{s_2}{k_I^+}\right)^2}.\end{aligned}$$

Any $\mu_j(s_j) \in [\mu_j(s_j)]$, $j = 1, 2$, satisfies Assumption 1; it also follows that there exist intervals for the kinetic coefficients, such that Assumption 4 is satisfied for any $\mu_j(s_j) \in [\mu_j(s_j)]$, $j = 1, 2$. Such intervals are for example the following:

$$\begin{aligned}[m_1] &= [1.2,\ 1.4], & [k_{s_1}] &= [6.5,\ 7.2],\\ [m_2] &= [0.64,\ 0.84], & [k_{s_2}] &= [9,\ 10.28],\\ [k_I] &= [15,\ 17].\end{aligned}$$

To simulate Assumption A5 we assume intervals for the coefficients k_j to be given, i. e. $k_j \in [k_j] = [k_j^-, k_j^+]$, $j = 1, 2, 3, 4$. Such numerical intervals are

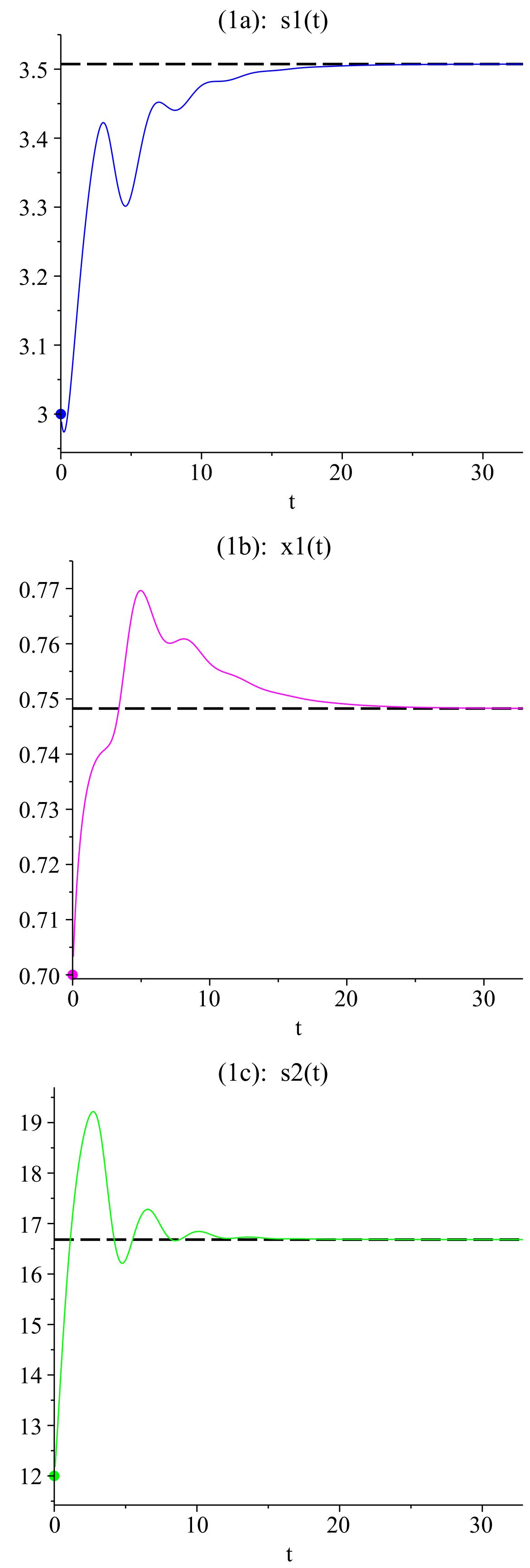
$$\begin{aligned}[k_1] &= [9.5, 11.5], & [k_2] &= [27.6, 29.6],\\ [k_3] &= [1064, 1084], & [k_4] &= [650, 700].\end{aligned}$$

All above intervals are chosen to enclose the numerical coefficients values derived by experimental measurements [1]; the values $\alpha = 0.5$, $s_1^i = 7.5$, $s_2^i = 75$ are also taken from [1]. Further, we assume that for any value $k_1 \in [k_1]$ and $k_2 \in [k_2]$ we have measurements $\frac{k_2}{k_1}s_1 + s_2$ for BOD, which is needed in Theorem 1.

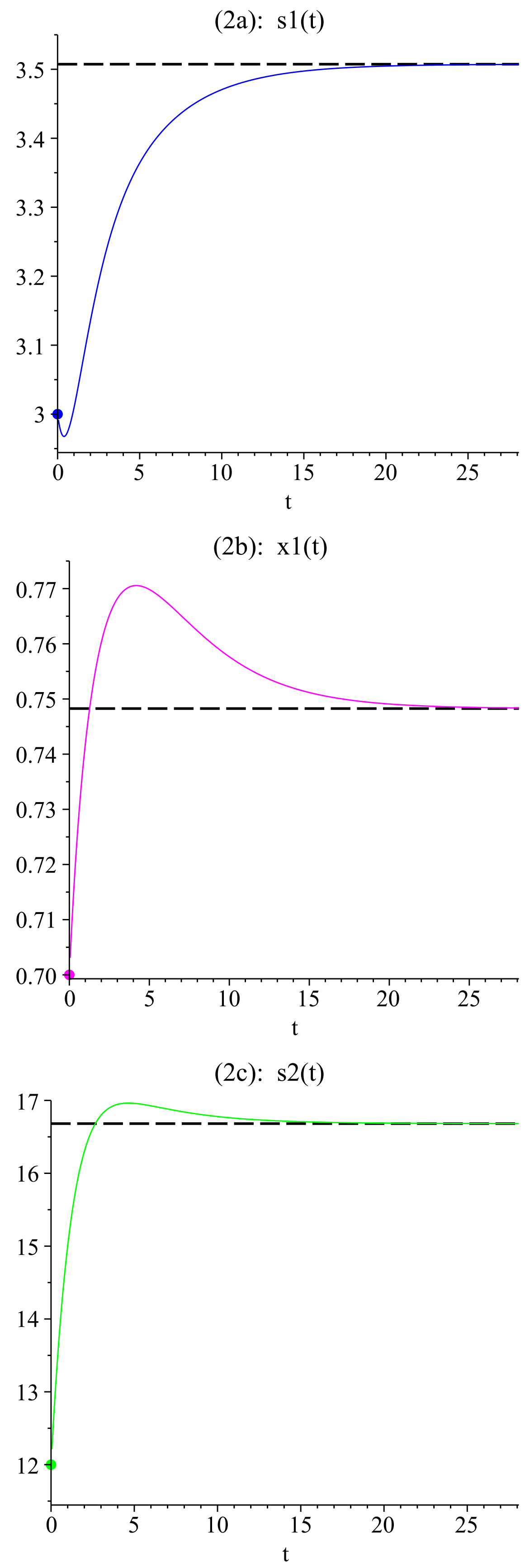
The simulation process is carried out in the following way. At the initial time $t_0 = 0$ we take random values for the coefficients from the corresponding intervals. To compare the effectiveness of the two stabilizing approaches we first take some operating point $\bar{s}$ (see (6)) and compute the equilibrium point $\bar{\zeta}$. By choosing an appropriate value for β from the corresponding interval (see (18)) we find an equilibrium point $\bar{p}$, such that $\bar{\zeta} = \bar{p}$. Note that the feedback laws $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ can be presented (see (5)) as $\kappa_1(t) = \beta(t) \cdot Q(t)$, $\kappa_2(t) = \beta \cdot Q(t)$. The difference lies in the choice of β: according to Theorem 1, $\beta(t)$ is a solution of the differential equation (9); according to Theorem 2, β is a positive constant within $\beta \in \left(\dfrac{k_3^+}{s^{i-} \cdot k_4^-}, +\infty\right)$.

The next figures (1a) to (1f) show time profiles of the phase variables $s_1(t)$, $x_1(t)$, $s_2(t)$, $x_2(t)$, as well as of $Q(t)$ and of the feedback $\kappa_1(t)$; figures (2a) to (2f) show the time evolution of the same phase variables, of $Q(t)$ and the feedback $\kappa_2(t)$. The horizontal dash-lines pass through the corresponding components of the equilibrium points $\bar{\zeta} = \bar{p}$ respectively, the values of $Q(\bar{s}_2, \bar{x}_2)$ and the feedbacks $\kappa_1(\bar{\zeta}, \bar{\beta})$ and $\kappa_2(\bar{p})$.

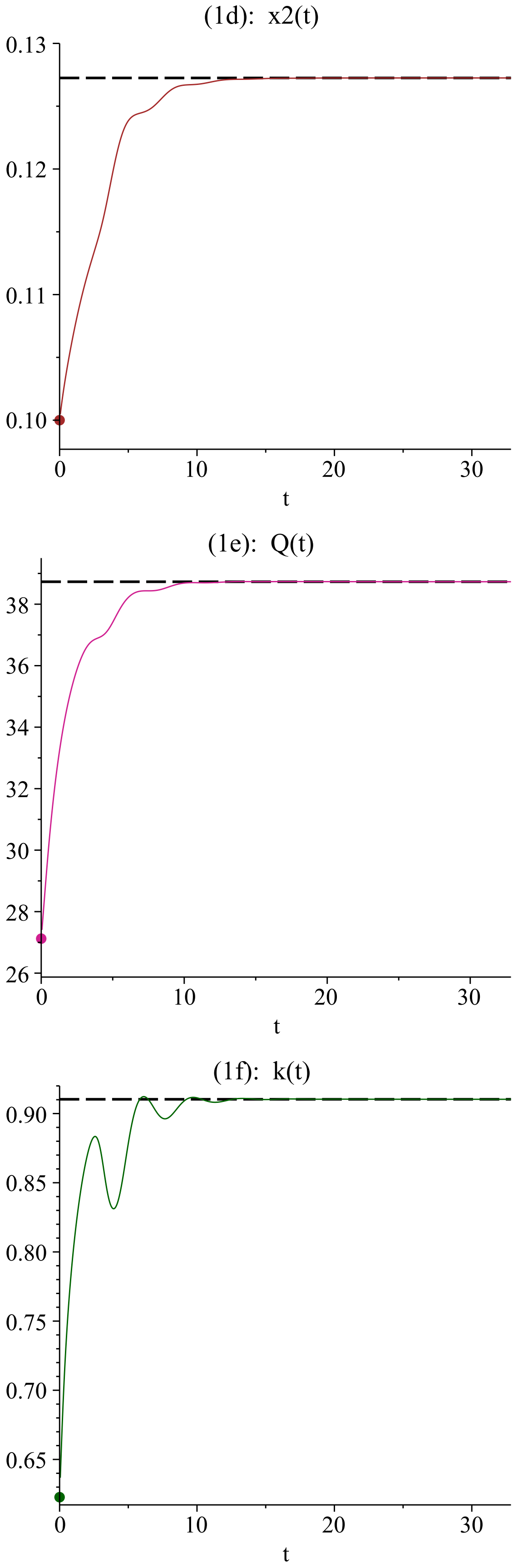
The numerical simulation and the graphic visualization are carried out in the computer algebra system *Maple 13*. A procedure for solving the closed-loop system of ordinary differential equations was designed; thereby the computations were stopped when the difference between the computed value of the solution and the value of the operating point became less than $\epsilon = 10^{-7}$.



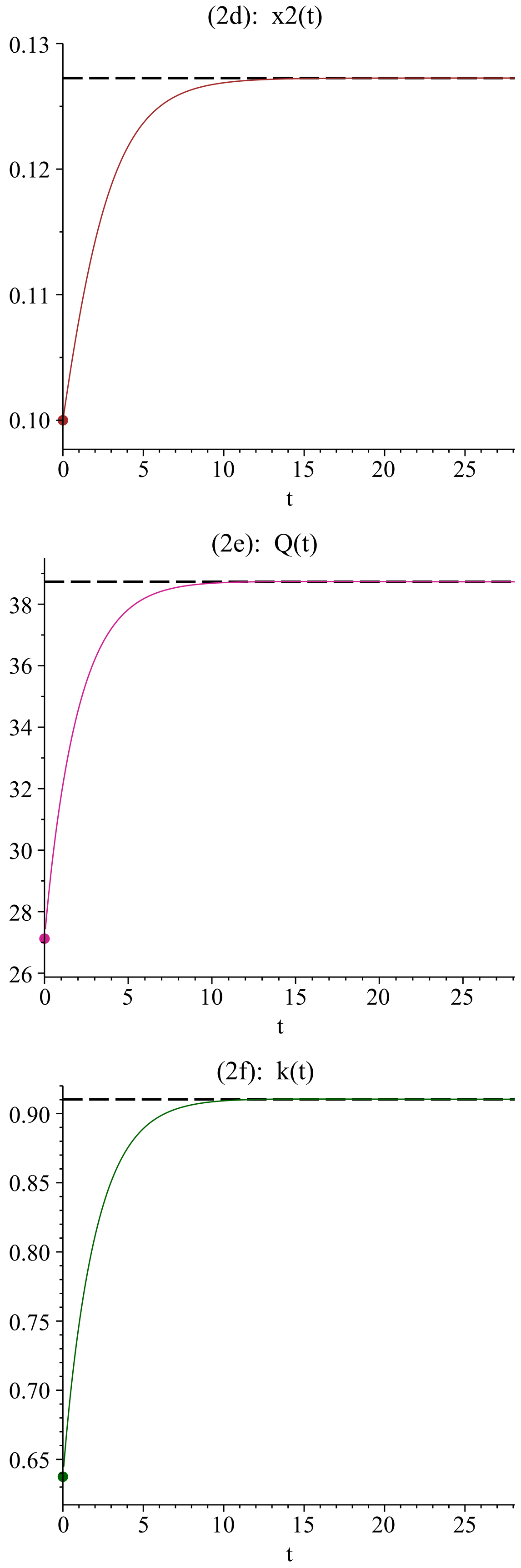
Time evolution of: (1a) $s_1(t)$, (1b) $x_1(t)$ and (1c) $s_2(t)$ using the adaptive asymptotic stabilization approach.



Time evolution of: (2a) $s_1(t)$, (2b) $x_1(t)$ and (2c) $s_2(t)$ using the non-adaptive asymptotic stabilization approach.



Time evolution of: (1d) $x_2(t)$, (1e) $Q(t)$ and (1f) $\kappa_1(t)$ using the adaptive asymptotic stabilization approach.



Time evolution of: (2d) $x_2(t)$, (2e) $Q(t)$ and (2f) $\kappa_2(t)$ using the non-adaptive asymptotic stabilization approach.

It is well seen from the figures that (i) both feedback laws stabilize globally the closed-loop system; (ii) the adaptive stabilization requires more time to approach the operating point than the non-adaptive stabilizer; (iii) there are oscillations of the solutions in the case of adaptive stabilization.

6 Conclusion

The present paper is devoted to the asymptotic stabilization of a four-dimensional nonlinear dynamic system, which models anaerobic degradation of organic wastes. Two approaches for global stabilization are presented. The first approach is based on adaptive feedback law; the second one uses a state feedback control law. To prove the stabilizabilty of the control system, explicit Lyapunov-like functions are constructed and a recent extension of the LaSalle invariance principle is used. The robustness of the proposed feedback laws is demonstrated in a computer simulation process, assuming that the model parameters are uncertain but bounded within compact intervals.

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