Stability Map of Fractional Order Time-Delay Systems

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Abstract: - In this paper, the stability robustness is considered for linear time invariant (LTI) fractional order systems with time delay against delay uncertainties. The complexity arises due to the exponential type transcendental terms and fractional order in their characteristic equation. We show that this procedure numerically reveals all possible stability regions exclusively in the space of the delay .Using the approach presented in this study, we can find all the locations where roots cross the imaginary axis. Finally, the concept of stability as a function of time delay is described for a general class of linear fractional order systems with multiple commensurate delays. Several numerical examples are provided to demonstrate the effectiveness of the proposed methodology.

Key-Words: - Fractional order systems, time-delay systems, characteristic equation, stability, Root-Locus.

1 Introduction

It is known that the presence of delay many cause poor performance and/or instability in a dynamical system. Therefore, time-delayed systems play significant roles in theoretical as well as practical fields; and this influence can be observed in numerous research articles written on various problems that involve this class of systems [1-6]. Fractional differential equations have gained considerable importance due to their application in various sciences, such as viscoelasticity, electroanalytical chemistry, electric conductance of biological systems, modeling of neurons, diffusion processes, damping laws, rheology, etc. Fractional order differential equation is represented in continuous-time domain by differential equations of non integer-order. Moreover, time delay is often present in real processes due to transportation of materials or energy. Therefore, most fractional systems may contain a delay term, such as fractional order neutral systems or some other fractional order delay systems (see [20,22] and the references cited in it). The characteristic function of a fractional-delay system involves exponential type transcendental terms, so a fractional delay system has in general an infinite number of characteristic roots. This makes the stability analysis of fractional-delay systems a challenging task. However, for fractional order dynamic systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. At the moment, direct check of the stability of fractional order systems using polynomial criteria (e.g., Routh's or Jury's type) is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopolynomial function of fractional powers of the complex variable s.

The researchers of [7] and [8] may be the pioneer to consider stability of the fractional order time delay system with single-delay. They have developed the Ruth-Hurwitz criteria for analyzing the stability of some special delay systems to those involve fractional power $\sqrt{s}$.

One of the important and basic things about each of the dynamical systems is the stability investigation. With respect to systems with delay, we'd like to study that how this stability property could behave, if we increased the time delay. It is known that an interesting phenomenon, namely stability windows, might happen. There has been a large effort to deal with this problem, as can be seen by the large quantity of articles dealing with it for the standard case; see. [9, 10,21,23], and many others. Recently, [11] has used numerical methods to investigate this subject in fractional order delay systems.

Bonnet and Partington introduced necessary and sufficient conditions for BIBO stability of the retarded fractional order delay systems and sufficient conditions for some neutral types [12]. Also, they studied necessary and sufficient conditions for H_∞ -stability (a weaker notation than BIBO stability) for a significant case of neutral type fractional order delay systems [13]. From the numerical analysis point of view, the effective numerical algorithms to check BIBO stability of fractional order delay systems have been discussed in [11] and [14]. In [15], a numerical method with complicated calculations based on the Cauchy's integral theory has been proposed for testing the stability of such systems. Some numerical techniques based on the Lambert W function have been presented in [16, 17] for the investigation of stability of these systems, where the rightmost characteristic root on the first Riemann sheet can be expressed explicitly in terms of Lambert W function.

In the present work, an exact method for the BIBO stability analysis of a large class of fractional order delay systems has been proposed, which is able to determine all the possible stability regions in the parametric space of these systems as a function of time delay. The main idea of this strategy has been adopted from [11], whose authors have studied the stability of integer order time-delayed systems. The rest of the present article has been organized as follows. Section 2 includes the necessary explanations and assumptions and also the stability test of these systems. In section 3, an exact method which expresses the stability regions as a function of delay has been presented. In section 4, some examples that demonstrate the effectiveness of the proposed approach have been described; and section 5 concludes the article.

2 Problem Formulation

Standard notation has been used throughout the article. The set of natural numbers is denoted by $\mathbb{N}$, whereas $\mathbb{N}_N$ denotes the set of its first N elements (i.e., $\mathbb{N}_N = \{1,\ldots,N\}$). $\mathbb{C}\left(\mathbb{C}^+,\mathbb{C}^-\right)$ is the set of complex numbers (with strictly positive, and strictly negative real parts), and $j=\sqrt{-1}$ is the imaginary unit . For $z\in\mathbb{C}$, $\bar{z}$ denotes its complex conjugate, and $\angle z$, $\Re(z)$ and $\Im(z)$ define the argument (taken here from $(-\pi,\pi]$), the real part and the imaginary part of z . $\mathbb{R}\left(\mathbb{R}^+,\mathbb{R}^-\right)$ denotes the set of real numbers (larger or equal to zero, smaller or equal to zero).

Consider a fractional order system with the following characteristic equation:

$$C\left(s,\tau\right)=\sum_{k=0}^{N} p_k\left(s^\alpha\right)e^{-ks\tau} \tag{1}$$

where parameter τ is non-negative, such that $\tau\in\mathbb{R}^+$; $p_k\left(s^\alpha\right)$ for $k\in\mathbb{N}_N$ are polynomials in s^α (where $\alpha\in(0,1)$). Note that the zeros of characteristic equation (1) are in fact the poles of the system under investigation. We find out from [14] that the transfer function of a system with a characteristic equation in the form of (1) will be H_∞ stable if, and only if, it doesn't have any pole at $\Re(s)\geq 0$ (in particular, no poles of fractional order at $s=0$).

For fractional order systems, if a auxiliary variable of $\varsigma=s^\alpha$ is used, a practical test for the evaluation of stability can be obtained. By applying this substitution in characteristic equation (1), the following relation is obtained:

$$C_\varsigma\left(\varsigma,\tau\right)=\sum_{k=0}^{N} p_k\left(\varsigma\right)e^{-k\tau\varsigma^{1/\alpha}} \tag{2}$$

For this new variable, the stability region of the original system is not expressed as the right half-plane, but as the region described below:

$$\left|\angle\varsigma\right|\leq\frac{\pi\alpha}{2} \tag{3}$$

with $\varsigma\in\mathbb{C}$, which the stable region has been displayed by shaded regions in Figure (1).

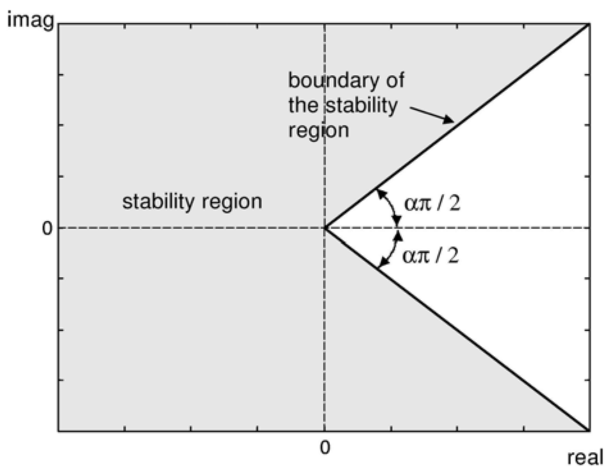

Fig. 1. The Stability Regions (Shaded) for linear fractional order systems

Note that under this transformation, the imaginary axis in the s-domain is mapped into the line

$$\angle\varsigma = \pm\frac{\pi\alpha}{2} \tag{4}$$

in the ς-domain, and therefore a solution $\varsigma^* = \left|\varsigma^*\right| \angle \pm \pi\alpha/2$ implies that the original system has a purely imaginary solution of the type

$$s^* = \pm j\left|\varsigma^*\right|^{\alpha^{-1}} \tag{5}$$

Let us assume that ; $s = \pm j\,\omega$ or in other words, $s = \omega\, e^{\pm j\pi/2}$ are the roots of characteristic equation (1) for a $\tau \in \mathbb{R}^+$. Then for the auxiliary variable, the roots are defined as follows:

$$\varsigma = s^{\alpha} = \omega^{\alpha}\, e^{\pm j\pi\alpha/2} \tag{6}$$

Therefore, with the auxiliary variable $\varsigma = s^{\alpha}$, there is a direct relation between the roots on the imaginary axis for the s-domain with the ones having argument $\pm\pi\alpha/2$ in the ς-domain.

3 Problem Solution

This is well known that one of the most important tools in the stability analysis of the systems is to use the location of the roots of system or root-locus method. About fractional-order delay systems, since there is exponential type transcendental terms in the characteristic equation and also because of having the fractional feature of the system, drawing the root-locus of the fractional delay systems makes the problem much more challenging compared to integer order systems. Some of the researchers have tried to present some algorithms to be able to draw the location of the roots. As an example, J.A. Tenreiro Machado in [19] and Fioravanti et al in [11]. But unfortunately, none of these algorithms could not properly display the root-locus of the fractional delay systems. We've tried to develop the existed algorithms, which are known in the resources, and also to draw the stability map and location of the roots correctly.

3.1 Crossing position

The main objective of this section is to present a new method for the evaluation of stability and determination of the unstable roots of a fractional order time delay system. A necessary and sufficient condition for the system to be asymptotically stable is that all the roots of the characteristic equation (1) lie in the left half of the complex plane.The proposed method eliminates the transcendental term of the characteristic equation without using any approximation or substitution and converts it into a equation without the transcendentality such that its real roots coincide with the imaginary roots of the characteristic equation exactly.

Based on the D-subdivision method, the number of unstable roots of a characteristic equation is invariant in some distinct regions of one-dimensional parameter space of time delay and the characteristic equation has at least one pair of purely imaginary roots at the boundary of these regions. After finding the boundaries and calculating the direction of imaginary roots variation, the number of unstable roots in each distinct interval is determined.

3.2 Single-Delay case

When there exists only a single delay in the system, the characteristic equation (1) becomes.

$$C\left(s^{\alpha},\tau\right) = p_0\left(s^{\alpha}\right) + p_1\left(s^{\alpha}\right)e^{-\tau s} \tag{7}$$

If for some finite $\tau, C\left(s^{\alpha},\tau\right) = 0$ has root on the imaginary axis at $s = j\,\omega_c$ (where subscript c refers to "crossing" the imaginary axis), then the equation $C\left((-s)^{\alpha},\tau\right) = 0$ must have the same root for the same value of τ because of the complex conjugate symmetry of roots. Therefore, looking for roots on the imaginary axis reduces to finding values of τ for which $C\left(s^{\alpha},\tau\right) = 0$ and $C\left((-s)^{\alpha},\tau\right) = 0$ have a common root. That is

$$\begin{aligned} C\left(s^{\alpha},\tau\right) &= p_0\left(s^{\alpha}\right) + p_1\left(s^{\alpha}\right)e^{-\tau s} = 0 \\ C\left((-s)^{\alpha},\tau\right) &= p_0\left((-s)^{\alpha}\right) + p_1\left((s)^{\alpha}\right)e^{\tau s} = 0 \end{aligned} \tag{8}$$

Let define variable $\overline{\varsigma}$ that is complex conjugate of auxiliary variable $\varsigma = \sqrt[\alpha]{\omega}e^{\,j\pi/2\alpha}$ as follows:

$$\varsigma = \sqrt[\alpha]{-s} = \sqrt[\alpha]{\omega} e^{-j\pi/2\alpha} = \left(e^{-j\pi/\alpha}\right)\varsigma \tag{9}$$

Where $\varsigma = \sqrt[\alpha]{\omega} e^{j\pi/2\alpha}$. Equation (8) can be written as follows

$$\begin{aligned} p_0\left(\varsigma^\alpha\right) + p_1\left(\varsigma\right) e^{-\tau\varsigma^\alpha} &= 0 \\ p_0\left(\bar{\varsigma}^\alpha\right) + p_1\left(\bar{\varsigma}^\alpha\right) e^{\tau\varsigma^\alpha} &= 0 \end{aligned} \tag{10}$$

By eliminating the exponential term in (10), we get the following polynomial:

$$p_0\left(\varsigma\right) p_0\left(\bar{\varsigma}\right) - p_1\left(\varsigma\right) p_1\left(\bar{\varsigma}\right) = 0 \tag{11}$$

Please note that transcendental characteristic equation with single delay given in (10) is now converted into a equation without transcendentality given by (11) and its positive real roots coincide with the imaginary roots of (7) exactly. The roots of this equation may easily be determined by standard methods. Depending on the roots of (11), the following situation may occur:

1. The equation of (10) does not have any positive real roots, which implies that the characteristic equation (6) does not have any roots on the $j\omega$-axis. In that case, the system is stable for all $\tau \geq 0$, indicating that the system is delay-independent stable.
2. The equation of (10) has at least one positive real root, which implies that the characteristic equation (6) has at least a pair complex eigenvalues on the $\tau \geq 0$-axis. In that case, the system is delay - dependent stable.

The roots of this equation may easily be determined by standard methods. For a positive real root ω_c, the corresponding value of delay margin τ can be easily obtained using (7) as:

$$\tau = \frac{1}{\omega_c}\left(\frac{\Im\left[p_0(\varsigma) \big/ p_1(\varsigma) \right]}{\Re\left[\left(-p_0(\varsigma)\right) \big/ p_1(\varsigma) \right]}\right)\Bigg|_{\varsigma=\sqrt[\alpha]{\omega}\, e^{\frac{j\pi}{2\alpha}}} + \frac{2k\pi}{\omega_c} \tag{12}$$

For the positive roots of (11), we also need to check if at $s = j\omega_c$, the root of characteristic equation (7) crosses the imaginary axis with increasing τ. This can be determined by the sign of $\Re\left[ds/d\tau\right]$. With these values in hand, we will be able to calculate the direction of crossing from the left half plane to the right one, which we will denote as a destabilizing crossing, or from the right to the left, meaning this is a stabilizing crossing. Notice that the use of the expressions destabilizing and stabilizing crossings means only that a pair of poles is crossing the imaginary axis in the defined direction, and not that the system is turning unstable or stable, respectively. For that, it is necessary to know the number of unstable poles before the crossings.

3.2 Commensurate-Delay case

The method given for the single-delay case could be easily extended to the stability analysis of the fractional order system with multiple commensurate time delays. The characteristic equation of such a system is given by (1). Similar to the single-delay case, if the characteristic equation (1) has a solution of $s = j\omega_c$ then $C\left((-s)^\alpha, \tau\right) = 0$ will have the same solution.

$$C\left((-s)^\alpha, \tau\right) = \sum_{k=0}^{N} p_k \left((-s)^\alpha\right) e^{ks\tau} \tag{13}$$

Characteristic equation (13) can be written in terms of the auxiliary parameter $\bar{\varsigma}$ as:

$$C\left(\bar{\varsigma}^\alpha, \tau\right) = \sum_{k=0}^{N} p_k \left(\bar{\varsigma}^\alpha\right) e^{k\varsigma^\alpha\tau} \tag{14}$$

Recall that in the single-delay case, the transcendental term and the time delay τ are eliminated. In this case, the purpose is the same. A recursive procedure should be developed to achieve that purpose. Therefore, let us define

$$\begin{aligned} &\sum_{k=0}^{n-1}\left[p_0\left(\bar{\varsigma}\right) p_k\left(\varsigma\right) - p_n\left(\varsigma\right) p_{n-k}\left(\bar{\varsigma}\right)\right] e^{k\varsigma^\alpha\tau} \\ &= C^{(1)}\left(\varsigma, \tau\right) = 0 \end{aligned} \tag{15}$$

Then, we have

$$\sum_{k=0}^{n-1}\left[p_0(\varsigma)p_k(\bar{\varsigma})-p_n(\bar{\varsigma})p_{n-k}(\varsigma)\right]e^{k\varsigma^\alpha\tau} = C^{(1)}(\bar{\varsigma},\tau)=0 \tag{16}$$

It should be observed from equations (15) and (16) that if $s = j\omega_c$ is the solution of equations (1) and (12) for some τ, then it must be a solution of the following augmented characteristic equations

$$\begin{aligned} C^{(1)}(\varsigma,\tau) &= \sum_{k=0}^{n-1} p_k^{(1)}(\varsigma)e^{k\varsigma^\alpha\tau}=0 \\ C^{(1)}(\bar{\varsigma},\tau) &= \sum_{k=0}^{n-1} p_k^{(1)}(\bar{\varsigma})e^{k\varsigma^\alpha\tau}=0 \end{aligned} \tag{17}$$

Where

$$p_k^{(1)}(\varsigma)=p_0^{(1)}(\bar{\varsigma})p_k^{(1)}(\varsigma)-p_n^{(1)}(\varsigma)p_{n-k}^{(1)}(\bar{\varsigma}) \tag{18}$$

Note that the characteristic equations (17) are of commensuracy degree of (n-1). We can easily repeat this procedure to eliminate commensuracy terms successively by defining a new polynomial

$$\begin{aligned} & p_k^{(r+1)}(\varsigma)= \\ & \quad p_0^{(r)}(\bar{\varsigma})p_0^{(r)}(\varsigma)-p_{n-r}^{(r)}(\varsigma)p_{n-r-k}^{(r)}(\bar{\varsigma}) \end{aligned} \tag{19}$$

and an augmented characteristic equation

$$C^{(r)}(\varsigma,\tau)=\sum_{k=0}^{n-r} p_k^{(r)}(\varsigma)e^{-k\varsigma^\alpha\tau}=0 \tag{20}$$

By repeating this procedure n times, we eliminate the highest degree of commensuracy terms and obtain the following augmented characteristic equation

$$C^{(n)}(\varsigma)=p_0^{(n)}(\varsigma)=0 \tag{21}$$

Where

$$\begin{aligned} & p_0^{(n)}(\varsigma)= \\ & p_0^{(n-1)}(\bar{\varsigma})p_0^{(n-1)}(\varsigma)-p_1^{(n-1)}(\varsigma)p_1^{(n-1)}(\bar{\varsigma})=0 \end{aligned} \tag{22}$$

It should be emphasized that that if $s = j\omega_c$ is the solution of (1) for some τ, then it is also a solution of (20) since the imaginary roots of the original characteristic equation (1) are preserved during the manipulations. If we substitute $\bar{\varsigma}=\left(e^{-j\pi/\alpha}\right)\varsigma$ and $\varsigma=\sqrt[\alpha]{\omega}e^{j\pi/2\alpha}$ in (22), we get the following equation in ω

$$\begin{aligned} D(\omega)=\Big(& p_0^{(n-1)}\left(e^{-j\pi/\alpha}\varsigma\right)p_0^{(n-1)}(\varsigma) \\ & -p_1^{(n-1)}(\varsigma)p_1^{(n-1)}\left(e^{-j\pi/\alpha}\varsigma\right)\Big)\Big|_{\varsigma=\sqrt[\alpha]{\omega}e^{j\pi/2\alpha}}=0 \end{aligned} \tag{23}$$

One can easily notice that (22) is the generalization of (11) and allows us to determine the imaginary roots of charactristic equation (1), if there exists any. The corresponding value of time delay is then computed by

$$\tau=\frac{1}{\omega_c}\left.\left(\frac{\Im\left[\dfrac{p_0^{(n-1)}(\varsigma)}{p_1^{(n-1)}(\varsigma)}\right]}{\Re\left[\dfrac{\left(-p_0^{(n-1)}(\varsigma)\right)}{p_1^{(n-1)}(\varsigma)}\right]}\right)\right|_{\varsigma=\sqrt[\alpha]{\omega}\,e^{\frac{j\pi}{2\alpha}}}+\frac{2k\pi}{\omega_c} \tag{24}$$

The whole ω values, for which $s = j\omega$ is a root of equation (1) for some non-negative delays, is defined as the crossing frequency set.

$$\Omega=\left\{\omega\in\mathbb{R}^+ \,\middle|\, C(s,\tau)=0,\ \text{for some}\ \ \tau\in\mathbb{R}^+\right\} \tag{25}$$

This class of systems exhibits only a finite number of possible imaginary characteristic roots for all $\tau\in\mathbb{R}^+$ at given frequencies. And this method detected all of them. Let us call this set

$$\{\omega_c\} = \{\omega_{c1}, \omega_{c2}, \ldots, \omega_{cn}\} \quad (26)$$

where subscript c refers to 'crossing' the imaginary axis. Furthermore to each ω_{cm}, $m = 1,\ldots,n$ correspond infinitely many, periodically spaced τ values. All this set

$$\{\tau_m\} = \{\tau_{m1}, \tau_{m2}, \ldots, \tau_{m\infty}\} \quad m = 1,\ldots,n \quad (27)$$

Where $\tau_{m,k+1} - \tau_{m,k} = 2\pi/\omega_m$ is the apparent period of repetition.

3.3 Direction of crossing

After the crossing points of characteristic equation (1) from the imaginary axis are obtained, the goal now is to determine whether each of these root crossings from the imaginary axis is a stabilizing cross or a destabilizing cross. As it was shown in [11], this is constant with respect to subsequential crossings , and therefore it is denoted as root tendency. Assume that (s,τ) is a simple root of $C(s,\tau)=0$. The root sensitivities associated with each purely imaginary characteristic root $j\omega$, with respect to τ is defined as

$$S_\tau^s\Big|_{s=\omega j} = \left.\frac{ds}{d\tau}\right|_{s=\omega j} = -\left.\frac{\partial C/\partial\tau}{\partial C/\partial s}\right|_{s=j\omega} = \left.\frac{s\sum_{k=1}^{N}\ell q_k\left(s^\alpha\right)e^{-\tau sk}}{p'\left(s^\alpha\right)+\sum_{k=1}^{N}\left(q_k'\left(s^\alpha\right)-\tau\ell q_k\left(s^\alpha\right)\right)e^{-\tau sk}}\right|_{s=j\omega} \quad (28)$$

Here, $p'\left(s^\alpha\right)$ and $q_\ell'\left(s^\alpha\right)$ denote the derivative of the polynomials $p\left(s^\alpha\right)$ and $q_\ell\left(s^\alpha\right)$ in s respectively. The root tendency is given by

It must be noted that the root tendency is independent of time delay τ. This implies that even though there is an infinite number of values of τ associated with each value of ω_c that make $C(j\omega_c,\tau)=0$.

$$\begin{aligned}
Root\ Tendency &= sign\left(\Re\left(S_\tau^s\Big|_{s=j\omega_{cm}}\right)\right)\\
&= sign\left(\Re\left(\frac{ds}{dv}\times\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\substack{s=j\omega\\ \varsigma=\varsigma_{cm}=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}}\\
&= sign\left(\Re\left(\varsigma^{\alpha-1}\times\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\substack{s=j\omega\\ \varsigma=\varsigma_{cm}=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}}\\
&= sign\left(\Re\left(je^{\frac{-j\pi}{2\alpha}}\times\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\substack{s=j\omega\\ \varsigma=\varsigma_{cm}=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}}\\
&= sign\left(\Re\left(\left[\sin\left(\frac{\pi}{2\alpha}\right)+j\cos\left(\frac{\pi}{2\alpha}\right)\right]\times\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\varsigma=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}\\
&= sign\left(\sin\left(\frac{\pi}{2\alpha}\right)\Re\left(\frac{d\varsigma}{d\tau}\right)-\cos\left(\frac{\pi}{2\alpha}\right)\Im\left(\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\varsigma=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}\\
&= sign\left(\tan\left(\frac{\pi}{2\alpha}\right)\Re\left(\frac{d\varsigma}{d\tau}\right)-\Im\left(\frac{d\varsigma}{d\tau}\right)\right)\Bigg|_{\varsigma=\varsigma_{cm}=\sqrt[\alpha]{\omega}\,e^{\pm\frac{\pi}{2\alpha}}}
\end{aligned} \quad (29)$$

If it is positive, then it is a destabilizing crossing, whereas if it is negative, this means a stabilizing crossing. In case the result is 0, a higher order analysis is needed, since this might be the case where the root just touches the imaginary axis and returns to its original half-plane. Notice that root tendency represents the direction of transition of the roots at $j\omega_c$ as τ increases from $\tau_{mk-\varepsilon}$ to $\tau_{mk+\varepsilon}$, $0<\varepsilon\le 1$.

$$sign\left(\Re\left(S_\tau^s\Big|_{\substack{s=j\omega\\ \tau}}\right)\right) \quad (30)$$

Is independent of k. Notice that for those given values, the exponential term

$$e^{-\tau sk} = e^{-j\theta\ell}e^{-j2\pi k\ell} = e^{-j\theta k} \quad (31)$$

is independent of ℓ.

Now having the cross points, their corresponding infinitely time delays, and root tendency of each time delay, the number of unstable roots in the regions subdivided by τ_{mk} can be calculated based on the D-subdivision method. To make a complete picture of the stability mapping for any fractional delay system, τ_{mk} is sorted from smallest to largest value. The number of unstable roots in each region between two successive time delays changes by $2RT$ if $\omega \neq 0$, and by RT if $\omega = 0$. Undoubtedly, after a specific value of time delay, the number of unstable roots cannot be zero and the calculation should be stopped.

4 Numerical Example

We present three example cases, which display all the features discussed in the text.

Example 1: This example has been taken from [7] and [15]. Consider the following linear time-invariant fractional order system with one delay:

$$C_1(s,\tau) = \left(\sqrt{s}\right)^3 - 1.5\left(\sqrt{s}\right)^2 + 4\left(\sqrt{s}\right) + 8 - 1.5\left(\sqrt{s}\right)^2 e^{-\tau s} \quad (32)$$

This system has a pair of poles ($s = \pm 8j$) on the imaginary axis for $\tau = 0$. A very involved calculation scheme based on Cauchy's integral has been used in [15] to show that this system is unstable for $\tau = 0.99$, and stable for $\tau = 1$. Our objective in this example is to find all the stability windows based on the method described in this article for this system.

Applying the first part of the algorithm, we can see that

$$\begin{aligned} \omega_1 &= 8 \to \tau = 0.7854k \\ \omega_2 &= 6.6248 \to \tau = 0.0499 + 0.9485k \end{aligned} \quad (33)$$

By applying the criterion expressed in the previous section, it is easy to find out that a destabilizing crossing of roots has occurred at $\tau = 0.7854k$ for $s = \pm 8j$ and a stabilizing crossing has taken place at $\tau = 0.0499 + 0.9485k$ for $s = \pm 6.6246j$, for all values of $k \in \{0,1,2,...\}$. Therefore, we will have 5 stability windows as follows: $0.0499 < \tau < 0.7854$, $0.9983 < \tau < 1.5708$, $1.9486 < \tau < 2.3562$, $2.8953 < \tau < 3.1416$ and $3.8437 < \tau < 3.9270$, which agree with the results presented by [7] and [11].

Note that at $\tau = 3.927$, an unstable pair of poles crosses toward the right half-plane, and before this unstable pole pair can turn to the left half-plane at $\tau = 4.7922$, another unstable pair of poles goes toward the right half-plane at $\tau = 4.7124$; and thus, the system cannot recover the stability.

In Table 1, the stability map of system defined via (32) is given. The number of unstable roots in each interval of unstable region has been determined as we

Table 1: Stability Regions (Shaded) for Example1

τ	$\omega\,(rad/s)$	Root Tendency	Number of unstable roots
0	8	+	2
0.0498	6.6248	-	
			0
0.7853	8	+	
			2
0.9938	6.6248	-	
			0
1.5707	8	+	
			2
1.9467	6.6248	-	
			0
2.3561	8	+	
			2
2.8952	6.6248	-	
			0
3.1415	8	+	
			2
3.8437	6.6248	-	
			0
3.9269	8	+	
			2
4.7123	8	+	
			4
4.7922	6.6248	-	
⋮	⋮		⋮

To get a better understanding of the properties of this system, its root-loci curve [19] has been plotted as a function of delay in Fig. 2. The color spectrum in the "color bar" indicates the selected τ; dark blue designates $\tau = 0$, and dark red is for $\tau = 3.9$. Since $\tau = 3.9$ is within the last stability window, we know in advance that the system will be stable for this amount of delay.

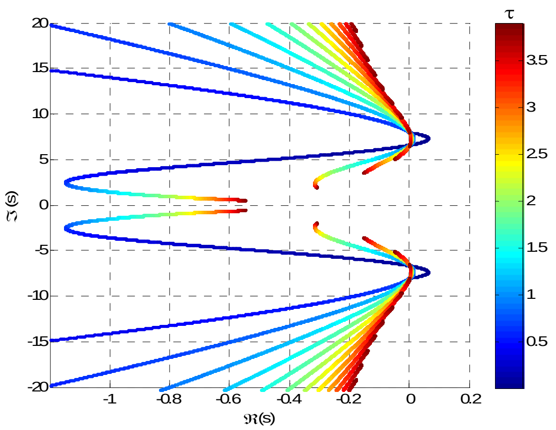

Fig. 2. Root-loci for $C_1(s,\tau)$ until τ=3.9

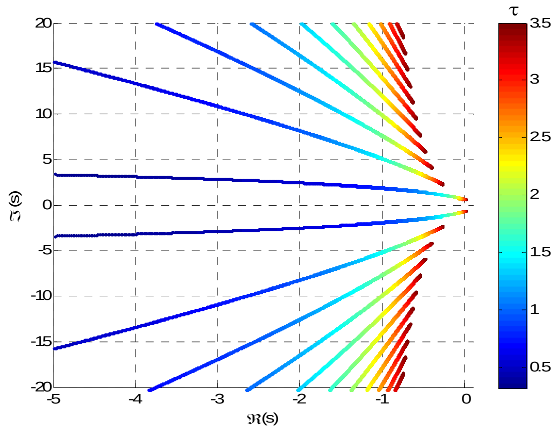

Fig. 3. Root-loci for $C_2(s,\tau)$ until τ=3.5

Example 2: This example comes from [18]. Consider the following fractional order system:

$$C_2(s,\tau)=2\sqrt{s}\left(\sqrt{s}+1\right)\left(\sqrt{s}+10\right) +5\left(\sqrt{s}+5\right)e^{-\tau s} \quad (34)$$

This system has no unstable pole for $\tau=0$ (in fact, it has no pole in the physical Riemann sheet). By applying the previously described method, it is realized that the crossing through the imaginary axis occurs at $\tau=3.2511598+9.9250867k$ and for $\omega=0.633061$, which this crossing is a destabilizing crossing, and it means that the only stability window for this system is $0<\tau<3.2511598$. In Table 2, the stability map of system defined via (34) is given. The number of unstable roots in each interval of unstable region has been determined as we

Table 2: Stability Regions (Shaded) for Example2

τ	$\omega\left(rad/s\right)$	Root Tendency	Number of unstable roots
0			
			0
3.25116	0.63306	+	
			2
13.17624	0.63306	+	
			4
23.10132	0.63306	+	
⋮	⋮		⋮

To get a better understanding of the properties of this system, its root-loci curve has been plotted as a function of delay in Fig. 3.

The color spectrum indicating the changes of τ from $\tau=0.322988$ (dark blue) to $\tau=3.5$ (dark red) has been illustrated in the "color bar".

Example 3: Consider the following fractional order system with 2 delays [15]:

$$C\left(s\right)=s^{5/6}+\left(s^{1/2}+s^{1/3}\right)e^{-0.5s}+e^{-s} \quad (35)$$

It has been demonstrated in [15] that this system is stable. Let's change system (35) to the following form [13]:

$$C_3\left(s,\tau\right)=s^{5/6}+\left(s^{1/2}+s^{1/3}\right)e^{-\tau s}+e^{-2\tau s} \quad (36)$$

Now we evaluate the stability of this system for all the values of τ and also study the stability for $\tau=0.5$.

By examining system (36) for $\tau=0$, we find out that for this value, it has no pole in the physical Riemann sheet and thus, the system is stable without delay. In view of relations (12) and (36) we obtain

$$\begin{aligned}\omega_1&=1\rightarrow\tau=2.3562+6.2832k\\ \omega_2&=1\rightarrow\tau=2.6180+6.2832k\end{aligned} \quad (37)$$

As is observed, the crossing of the imaginary axis for $\tau=2.3562+6.2832k$ and $\tau=2.6180+6.2832k$ occurs at $s=\pm j$, and both of these are destabilizing crossings; and since the system is stable for $\tau=0$, the only stability window for this system is $0\le\tau<2.3562$; and since $\tau=0.5$ falls within this window, we can be sure that the original system $C\left(s\right)$ is stable.

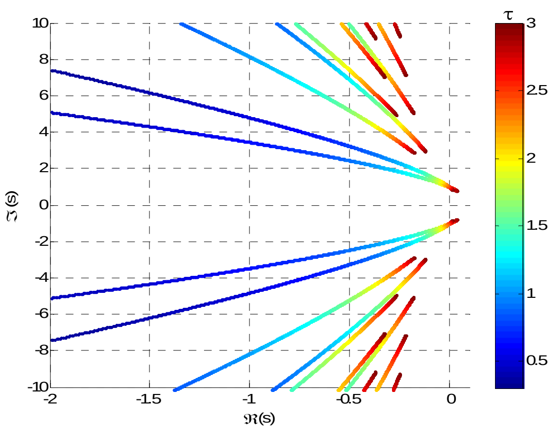

Fig. 4. Root-loci for $C_3(s,\tau)$ from τ=0.1 until τ=3

The obtained stability window matches the results presented in [15] and [11], which have used the numerical method to analyze this system.

In Fig. 4, the root-loci curve of this system for the changes of τ from $\tau = 0.1$ (dark blue) to $\tau = 3$ (dark red) has been presented for a better perception of the system.

5 Conclusion

In this paper, a new method for calculating stability windows and location of the unstable poles is proposed for a large class of fractional order time-delay systems. As the main advantages, we just deal with polynomials of the same order as that of the original system. and the use of auxiliary variable ς, the crossing points through the imaginary axis, and their direction of crossing, were determined. Then, system stability was expressed as a function of delay, based on the information obtained from the system. According to the infinitely countable time delays corresponding to each crossing point, the parametric space of τ is discretized to investigate stability in each interval. The number of unstable roots can be calculated with root tendency of each crossing point on the interval boundaries. Based on the proposed method, an upper bound for time delay is determined so that the system would not be stable any more for larger time delays. Finally, several examples were presented to highlight the proposed approach.

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