The Statistics of the Κ-μ-g Random Variables

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Abstract: - In this paper, the analysis of the κ-μ-g random variable is presented. The κ-μ-g random variable was obtained from κ-μ random variable whose power follows Gamma distribution. The closed form expressions for probability density function (PDF) and cumulative distribution function (CDF) of κ-μ-g random variable are determined. Then, PDFs of the moment of *n*-th order of κ-μ-g random variable, maximum, minimum, ratio and product of two κ-μ-g random variables are derived in the closed form. This analysis is important for the modelling of the wireless mobile communication systems in the presence of κ-μ-g short term fading, Gamma shadowing and κ-μ-g cochannel interference.

Key-Words: κ-μ-g distribution; random variable; probability density function; maximum; minimum; moments; product; ratio; shadowing; short term fading

1 Introduction

The characteristics of κ-μ-g random variable, as important performance of a compound fading model, are investigated in [1], [2]. Distribution of maximum and minimum is derived in [1]. The closed form expressions for probability density function (PDF) and cumulative distribution function (CDF) of κ-μ-g random variable are determined in [2]. Then, PDFs of the moment of *n*-th order of κ-μ-g random variable, the ratio and product of two κ-μ-g random variables are calculated in the closed form [2]. Using these expressions, performance analysis of wireless communication systems in the presence of κ-μ-g multipath fading and κ-μ-g cochannel interference can be made becuase this distribution describes composite fading model with gamma shadowed κ−μ multipath fading. The distribution κ−μ/gamma corresponds to a physical fading model [2]. This composite distribution is based on κ−μ generalized shadowed multipath fading model.

Namely, in wireless communications systems fading causes fluctuations of received signal envelope over time [4], [5]. There are some different statistical models which can describe fading in wireless communication channels. The most often used are Rayleigh, Rician, Nakagami-m, Weibull, Hoyt, α−μ, κ−μ, η−μ, α−κ−μ , α−η−μ. We will analyze here the κ−μ−g distribution of fading.

The κ-μ-g distribution has three parameters [2]. The parameter κ is Rician factor. It is defined as the ratio of the power of direct component of signal and the power of the scattered components. The κ-μ-g fading is less severe for higher values of factor κ. The parameter μ is tied to the number of clusters in propagation environment and the κ-μ-g multipath fading is less severe for bigger values of parameter μ. The parameter *c* is severity of Gamma shadowing and Gamma long term fading is less severe for higher values of parameter *c*. The κ-μ-g distribution is general distribution and some distributions can be derived from this distribution as special cases [2].

The statistics of two random variables are very important for performance analysis of wireless mobile communication systems. The ratio of two random variables can be used in performance analysis of wireless systems operating over multipath fading channel in the presence of cochannel interference. In cellular radio interference limited environment, the ratio of signal envelope and interference envelope is important system performance [6].

Also, the ratios of random variables can be applied in performance analysis of wireless mobile communication systems which work over multipath fading environment in the presence of cochannel interference suffered to multipath fading [7]- [8].

The statistics of two random variables are very important for performance analyzing of wireless mobile communication radio systems. Sum of two random variables can be used for calculation the outage probability of wireless system with equal gain combining (EGC) receiver with two inputs operating over short term fading channel. EGC receiver is used for reduction multipath fading effects on system performance [9].

The statistics of maximum of two random variables is often used for calculating the bit error probability (BER) of wireless communication system with selection combining (SC) receiver with two inputs, working over short term fading channel. The SC receiver output signal is equal to the maximum of signals at its inputs [10].

The statistics of minimum of two random variables is applicable in analysis of performance of wireless relay communication systems with two sections. Under determined conditions, signal envelope at output of relay system can be expressed as product of signal envelope at each section. Cumulative distribution of minimum of two random variables is used for calculation the outage probability of relay system with two sections.

In [11], the PDF of minimum of ratios of random variables is presented. It can be used to derive the expression for moments-generating function to evaluate lower bounds for the average bit error probability, as another important system performance measure, for different modulation schemes.

The statistics of ratio of two random variables can be used in performance analysis of wireless communication systems operating over short term fading channel in the presence of cochannel interference [12]. In cellular radio interference limited environment, the ratio of signal envelope and interference envelope is important system performance. The statistics of product of two random variables has application in performance analysis of wireless relay communication systems with two sections [13]. Under determined conditions, signal envelope at output of relay system can be expressed as product of signal envelope at each section [14].

In this paper, the statistics of two κ-μ-g random variables are investigated. The closed form expressions for probability density function (PDF) and cumulative distribution function (CDF) of κ-μ-g random variable are determined. Then, PDFs of the moment of *n*-th order of κ-μ-g random variable, maximum, minimum, ratio and product of two κ-μ-g random variables are derived in the closed form. The results obtained in this work can be applied in performance analysis of wireless mobile communication systems in the presence of κ-μ-g small scale fading.

This paper presents an overview of the performance of κ-μ-g random variable determined previously. It is organized as follows: in the second section, the κ-μ-g random variable is defined and the main quantities (PDF, CDF, moments, maximum, minimum, product and ratio) are calculated. In the third section, the numerical results are plotted and parameters influence is examined. Last section concludes the paper and gives some final remarks.

2 The Statistics of the κ-μ-g Random Variable

2.1 Probability Density Function of κ-μ-g Random Variable

The κ-μ-g distribution describes signal envelope in Gamma shadowed κ-μ multipath fading environment. The probability density function of κ-μ-g random variable is determined by integration of conditional κ-μ distribution [2]:

$$p_x\left(x\right)=\int_0^\infty d\Omega\ p_y\left(x/\Omega\right)d\Omega \tag{1}$$

where:

$$p_y\left(y/\Omega\right)=\frac{2\mu\left(k+1\right)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\,\Omega^{\frac{\mu+1}{2}}}y^{\mu}\cdot e^{-\frac{\mu(k+1)}{\Omega}y^2}\cdot I_{\mu-1}\left(2\mu_1\frac{\sqrt{k\left(k+1\right)}}{\Omega}y\right)=$$

$$=\frac{2\mu\left(k+1\right)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\,\Omega^{\frac{\mu+1}{2}}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{\frac{k\left(k+1\right)}{\Omega}}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma\left(i_1+\mu\right)}\cdot$$

$$\cdot y^{2i_1+2\mu-1}e^{-\frac{\mu(k+1)}{\Omega}y^2},\ y\geq 0 \tag{2}$$

and:

$$p_\Omega\left(\Omega\right)=\frac{1}{\Gamma\left(c\right)\beta^c}\Omega^{c-1}e^{-\frac{1}{\beta}\Omega},\ \Omega\geq 0, \tag{3}$$

Here, κ is Rician factor, μ is the number of clusters in propagation environment, $I_n(x)$ is modified Bessel function of the first kind and order n [15], $\Gamma(n)$ is (complete) gamma function, Ω is the signal envelope average power with Gamma distribution, $\beta=\overline{\Omega^2}$, c is Gamma fading severity parameter.

PDF of the κ-μ-g random variable may be written in the form [2]:

$$p_x(x)=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\Omega^{\frac{\mu+1}{2}}}\cdot$$

$$\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot x^{2i_1+2\mu-1}\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}}\cdot K_{c-i_1-\mu}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right).\quad(4)$$

$K_n(x)$ is modified Bessel function of the second kind [16, eq. (3.471.9)].

2.2 Cumulative distribution function of κ-μ-g Random Variable

The cumulative distribution function (CDF) of x is [2]:

$$F_x(x)=\int_0^x dt\ p_x(t)=$$

$$=\frac{\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{i_1+\mu}x^{2i_1+2\mu}\cdot\sum_{j_1=0}^{\infty}\frac{1}{(i_1+\mu+1)(j_1)}\cdot\left(\mu(k+1)x^2\right)^{j_1}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}-\frac{j_1}{2}}\cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)\quad(5)$$

2.3 Moment of n-th Order of the κ-μ-g Random Variable

The moment of n-th order of κ-μ-g random variable is [2]:

$$m_n=\overline{x^n}=\int_0^{\infty}dx\,x^n p_x(x)=$$

$$=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{2}\frac{1}{\left(\mu(k+1)\right)^{i_1+\mu+\frac{n}{2}}}\Gamma\left(i_1+\mu+\frac{n}{2}\right)\cdot\beta^{c+\frac{n}{2}}\Gamma\left(c+\frac{n}{2}\right).\quad(6)$$

2.4 Maximum of Two κ-μ-g Random Variables

The maximum of two κ-μ-g random variables x_1 and x_2 is:

$$x=\max(x_1,x_2)\quad(7)$$

The probability density function of x is:

$$p_x(x)=p_{x_1}(x)F_{x_2}(x)+p_{x_2}(x)F_{x_1}(x)=\frac{2m(k+1)^{\frac{m+1}{2}}}{k^{\frac{m-1}{2}}e^{km}}\times$$

$$\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot x^{2i_1+2\mu-1}\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}}\cdot K_{c-i_1-\mu}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)$$

$$\cdot\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_2=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_2+\mu-1}\frac{1}{i_2!\Gamma(i_2+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{i_2+\mu}x^{2i_2+2\mu}\sum_{j_1=0}^{\infty}\frac{1}{(i_2+\mu+1)(j_1)}\left(\mu(k+1)x^2\right)^{j_1}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_2}{2}-\frac{\mu}{2}-\frac{j_1}{2}}\cdot K_{c-i_2-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)\quad(8)$$

Cumulative distribution function (CDF) of the maximum of two κ-μ-g random variables x_1 and x_2 is:

$$F_x(x)=F_{x_1}(x)=F_{x_2}(x)=$$

$$=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{i_1+\mu}x^{2i_1+2\mu}\sum_{j_1=0}^{\infty}\frac{1}{(i_1+\mu+1)(j_1)}\left(\mu(k+1)x^2\right)^{j_1}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_2}{2}-\frac{\mu}{2}-\frac{j_1}{2}}\cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right).\quad(9)$$

It is important to derive the distribution of the maximum of random variables because SC receiver output signal is equal to the signal with maximal envelope if the noise is equally distributed at its inputs.

2.5 Minimum of Two κ-μ-g Random Variables

The minimum of two κ-μ-g random variables x_1 and x_2 is:

$$x=\min(x_1,x_2)\quad(10)$$

The cumulative distribution function of x is:

$$F_x(x)=1-\left(1-F_{x_1}(x)\right)\left(1-F_{x2}(x)\right)=$$

$$= 1- \left(1- F_x(x)\right)\left(1- F_x(x)\right) = 1- \left(1- F_x(x)\right)^2 =$$

$$= 1-\left(1-\frac{\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\right.$$

$$\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{i_1+\mu}x^{2i_1+2\mu}\cdot\sum_{j_1=0}^{\infty}\frac{1}{(i_1+\mu+1)(j_1)}\cdot\left(\mu(k+1)x^2\right)^{j_1}\cdot$$

$$\left.\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}-\frac{j_1}{2}}\cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)\right)^2 \quad (11)$$

The probability density function of minimum can be used to study the outage probability as an important multi-hop system performance measure. Various results complement the proposed mathematical analysis.

2.6 Ratio of Two κ-μ-g Random Variables

The distribution of the ratio of random variables is important in statistical analysis in a number of different fields of science and also in analysis of wireless communication systems in the presence of fading [17]. Videlicet, the random variable in nominator presents the desired signal envelope while the random variable in denominator presents the interference signal envelope.

Conditional κ-μ random variable is:

$$p_{x_1}(x_1/\Omega_1)=\frac{2\mu_1(\kappa_1+1)^{\frac{\mu_1+1}{2}}}{\kappa_1^{\frac{\mu_1-1}{2}}e^{\kappa_1\mu_1}\Omega_1^{\frac{\mu_1+1}{2}}}\cdot$$

$$\cdot\sum_{i=0}^{\infty}\left(\mu_1\sqrt{\frac{\kappa_1(\kappa_1+1)}{\Omega_1}}\right)^{2i_1+\mu_1-1}\frac{1}{i_1!\Gamma(i_1+\mu_1)}\cdot$$

$$\cdot x_1^{2i_1+2\mu_1-1}\cdot e^{-\frac{\mu_1(\kappa_1+1)}{\Omega_1}x_1^2},\ x_1\ge 0 \quad (12)$$

The probability density function of Ω_1 is:

$$p_{\Omega_1}(\Omega_1)=\frac{1}{\Gamma(c_1)\beta_1^{c_1}}\Omega_1^{c_1-1}e^{-\frac{1}{\beta_1}\Omega_1},\ \Omega_1\ge 0 \quad (13)$$

By averaging we have:

$$p_{x_1}(x_1)=\int_0^{\infty}d\Omega_1\cdot p_{x_1}(x_1/\Omega_1)p_{\Omega_1}(\Omega_1)=$$

$$=\frac{2\mu_1(\kappa_1+1)^{\frac{\mu_1+1}{2}}}{\kappa_1^{\frac{\mu_1+1}{2}}e^{\kappa_1\mu_1}\Omega_1^{\frac{\mu_1+1}{2}}}\cdot$$

$$\cdot\sum_{i=0}^{\infty}\left(\mu_1\sqrt{\kappa_1(\kappa_1+1)}\right)^{2i_1+\mu_1-1}\frac{1}{i_1!\Gamma(i_1+\mu_1)}\cdot$$

$$\cdot x_1^{2i_1+2\mu_1-1}\cdot\frac{1}{\Gamma(c_1)\beta_1^{c_1}}$$

$$\int_0^{\infty}d\Omega_1\,\Omega_1^{-\frac{\mu_1+1}{2}-i_1-\frac{\mu_1}{2}+\frac{1}{2}+c_1-1}e^{-\frac{\mu_1(\kappa_1+1)}{\Omega_1}x_1^2-\frac{1}{\beta_1}\Omega_1} \quad (13)$$

Conditional probability density of x_2 is:

$$p_{x_2}(x_2/\Omega_2)=\frac{2\mu_2(\kappa_2+1)^{\frac{\mu_2+1}{2}}}{\kappa_2^{\frac{\mu_1+1}{2}}e^{\kappa_2\mu_2}\Omega_2^{\frac{\mu_2+1}{2}}}\cdot$$

$$\cdot\sum_{i=0}^{\infty}\left(\mu_2\sqrt{\frac{\kappa_2(\kappa_2+1)}{\Omega_2}}\right)^{2i_2+\mu_2-1}\frac{1}{i_2!\Gamma(i_2+\mu_2)}\cdot$$

$$\cdot x_2^{2i_2+2\mu_2-1}\cdot e^{-\frac{\mu_2(\kappa_2+1)}{\Omega_2}x_1^2},\ x_2\ge 0 \quad (14)$$

The probability density function of Ω_2 is:

$$p_{\Omega_2}(\Omega_2)=\frac{1}{\Gamma(c_2)\beta_2^{c_1}}\Omega_2^{c_2-1}e^{-\frac{1}{\beta_2}\Omega_2},\ \Omega_2\ge 0 \quad (15)$$

By averaging, it is obtained:

$$p_{x_2}(x_2)=\int_0^{\infty}d\Omega_2\cdot p_{x_2}(x_2/\Omega_2)p_{\Omega_2}(\Omega_2)=$$

$$=\frac{2\mu_2(\kappa_2+1)^{\frac{\mu_2+1}{2}}}{\kappa_2^{\frac{\mu_2-1}{2}}e^{\kappa_2\mu_2}}\cdot$$

$$\cdot\sum_{i_2=0}^{\infty}\left(\mu_2\sqrt{\kappa_2(\kappa_2+1)}\right)^{2i_2+\mu_2-1}\frac{1}{i_2!\Gamma(i_2+\mu_2)}\cdot$$

$$\cdot x_2^{2i_2+2\mu_2-1}\cdot\frac{1}{\Gamma(c_2)\beta_2^{c_2}}$$

$$\int_0^{\infty}d\Omega_2\,\Omega_2^{-\frac{\mu_2+1}{2}-i_2-\frac{\mu_2}{2}+\frac{1}{2}+c_2-1}e^{-\frac{\mu_2(\kappa_2+1)}{\Omega_2}x_2^2-\frac{1}{\beta_2}\Omega_2} \quad (16)$$

The ratio of x_1 and x_2 is:

$$x = \frac{x_1}{x_2}, \quad x_1 = x \cdot x_2. \tag{17}$$

Probability density function of *x* is:

$$p_x(x) = \int_0^\infty dx_2 \cdot x_2 \, p_{x_1}(x x_2) p_{x_2}(x_2) =$$

$$= \frac{2\mu_1 (\kappa_1+1)^{\frac{\mu_1+1}{2}}}{\kappa_1^{\frac{\mu_1+1}{2}} e^{\kappa_1\mu_1} \Omega_1^{\frac{\mu_1+1}{2}}} \cdot$$

$$\cdot \sum_{i=0}^{\infty} \left(\mu_1 \sqrt{\kappa_1(\kappa_1+1)}\right)^{2i_1+\mu_1-1} \frac{1}{i_1! \Gamma(i_1+\mu_1)} \cdot$$

$$\cdot \frac{2\mu_2 (\kappa_2+1)^{\frac{\mu_2+1}{2}}}{\kappa_2^{\frac{\mu_2-1}{2}} e^{\kappa_2\mu_2}} \cdot$$

$$\cdot \sum_{i_2=0}^{\infty} \left(\mu_2 \sqrt{\kappa_2(\kappa_2+1)}\right)^{2i_2+\mu_2-1} \frac{1}{i_2! \Gamma(i_2+\mu_2)} \cdot$$

$$\cdot \frac{1}{\Gamma(c_1)\Gamma(c_2)} \cdot \frac{1}{\beta_1^{c_1} \beta_2^{c_2}} x_1^{2i_1+2\mu_1-1} \cdot$$

$$\int_0^\infty d\Omega_1 \, \Omega_1^{-\mu_1-i_1+c_1-1} e^{-\frac{1}{\beta_1}\Omega_1} \int_0^\infty d\Omega_2 \, \Omega_2^{-\mu_2-i_2+c_2-1} e^{-\frac{1}{\beta_2}\Omega_2}$$

$$\cdot \int_0^\infty dx_2 \cdot x_2^{2i_1+2\mu_1+2i_2+2\mu_2-1}$$

$$e^{-x_2^2 \frac{\mu_1(\kappa_1+1)\Omega_2 x^2 + \mu_2(\kappa_2+1)\Omega_1}{\Omega_1\Omega_2}} = \frac{2\mu_1 (\kappa_1+1)^{\frac{\mu_1+1}{2}}}{\kappa_1^{\frac{\mu_1+1}{2}} e^{\kappa_1\mu_1} \Omega_1^{\frac{\mu_1+1}{2}}} \cdot$$

$$\cdot \sum_{i=0}^{\infty} \left(\mu_1 \sqrt{\kappa_1(\kappa_1+1)}\right)^{2i_1+\mu_1-1} \frac{1}{i_1! \Gamma(i_1+\mu_1)} \cdot$$

$$\cdot \frac{2\mu_2 (\kappa_2+1)^{\frac{\mu_2+1}{2}}}{\kappa_2^{\frac{\mu_2-1}{2}} e^{\kappa_2\mu_2}} \cdot$$

$$\cdot \sum_{i_2=0}^{\infty} \left(\mu_2 \sqrt{\kappa_2(\kappa_2+1)}\right)^{2i_2+\mu_2-1} \frac{1}{i_2! \Gamma(i_2+\mu_2)} \cdot$$

$$\cdot \frac{1}{\Gamma(c_1)\Gamma(c_2)} \cdot \frac{1}{\beta_1^{c_1} \beta_2^{c_2}} x_1^{2i_1+2\mu_1-1} \cdot$$

$$\cdot \frac{1}{2} \Gamma(i_1+\mu_1+i_2+\mu_2)$$

$$\cdot \int_0^\infty d\Omega_1 \, \Omega_1^{-\mu_1-i_1+c_1-1+i_1+\mu_1+i_2+\mu_2} e^{-\frac{1}{\beta_1}\Omega_1}$$

$$\cdot \int_0^\infty d\Omega_2 \, \Omega_2^{-\mu_2-i_2+c_2-1+i_1+\mu_1+i_2+\mu_2} e^{-\frac{1}{\beta_2}\Omega_2}$$

$$\frac{1}{\left(\mu_1(\kappa_1+1)x^2\Omega_2 + \mu_2(\kappa_2+1)\Omega_1\right)^{i_1+\mu_1+i_2+\mu_2}} \tag{18}$$

Let the last double integral signed by *J* [19]:

$$J = \int_0^\infty d\Omega_1 \, \Omega_1^{-\mu_1-i_1+c_1-1+i_1+\mu_1+i_2+\mu_2} e^{-\frac{1}{\beta_1}\Omega_1}$$

$$\cdot \int_0^\infty d\Omega_2 \, \Omega_2^{-\mu_2-i_2+c_2-1+i_1+\mu_1+i_2+\mu_2} e^{-\frac{1}{\beta_2}\Omega_2}$$

$$\frac{1}{\left(\mu_1(\kappa_1+1)x^2\Omega_2 + \mu_2(\kappa_2+1)\Omega_1\right)^{i_1+\mu_1+i_2+\mu_2}} \tag{19}$$

Its solving is given in the Appendix.

By solving this integral, PDF of the ratio of two κ-μ-g random variables x_1 and x_2 is:

$$J = \beta_2^{c_1+c_2} \frac{1}{\left(\mu_1(\kappa_1+1)x^2\right)^{c_1+i_2+\mu_2}} \left(\mu_2(\kappa_2+1)\right)^{c_1-i_1-\mu_1}$$

$$\cdot \Gamma(c_2+i_1+\mu_1) \cdot \frac{\Gamma(c_1+c_2)\Gamma(c_1+i_2+\mu_2)}{\Gamma(c_1+c_2+i_1+i_2+\mu_1+\mu_2)}$$

$$ {}_2F_1\left(c_1+c_2, c_1+i_2+\mu_2, c_1+c_2+i_1+i_2+\mu_1+\mu_2; 1-\frac{\beta_2\mu_2(\kappa_2+1)}{\beta_1\mu_1(\kappa_1+1)x^2}\right) \tag{20}$$

By substituting (20) into (18), the PDF of the ratio of two κ-μ-g random variables is:

$$p_x(x) = \frac{2\mu_1 (\kappa_1+1)^{\frac{\mu_1+1}{2}}}{\kappa_1^{\frac{\mu_1+1}{2}} e^{\kappa_1\mu_1} \Omega_1^{\frac{\mu_1+1}{2}}} \cdot$$

$$\cdot \sum_{i=0}^{\infty} \left(\mu_1 \sqrt{\kappa_1(\kappa_1+1)}\right)^{2i_1+\mu_1-1} \frac{1}{i_1! \Gamma(i_1+\mu_1)} \cdot$$

$$\cdot \frac{2\mu_2 (\kappa_2+1)^{\frac{\mu_2+1}{2}}}{\kappa_2^{\frac{\mu_2-1}{2}} e^{\kappa_2\mu_2}} \cdot$$

$$\cdot \sum_{i_2=0}^{\infty} \left(\mu_2 \sqrt{\kappa_2(\kappa_2+1)}\right)^{2i_2+\mu_2-1} \frac{1}{i_2! \Gamma(i_2+\mu_2)} \cdot$$

$$\cdot \frac{1}{\Gamma(c_1)\Gamma(c_2)} \cdot \frac{1}{\beta_1^{c_1}\beta_2^{c_2}} x_1^{2i_1+2\mu_1-1} \cdot$$

$$\cdot \frac{1}{2}\Gamma(i_1+\mu_1+i_2+\mu_2)$$

$$\cdot \beta_2^{c_1+c_2} \frac{1}{\left(\mu_1(\kappa_1+1)x^2\right)^{c_1+i_2+\mu_2}} \cdot \left(\mu_2(\kappa_2+1)\right)^{c_1-i_1-\mu_1}$$

$$\cdot \Gamma(c_2+i_1+\mu_1) \cdot \frac{\Gamma(c_1+c_2)\Gamma(c_1+i_2+\mu_2)}{\Gamma(c_1+c_2+i_1+i_2+\mu_1+\mu_2)}$$

$${}_2F_1\left(c_1+c_2, c_1+i_2+\mu_2, c_1+c_2+i_1+i_2+\mu_1+\mu_2; 1-\frac{\beta_2\mu_2(\kappa_2+1)}{\beta_1\mu_1(\kappa_1+1)x^2}\right). \tag{21}$$

2.7 Product of Two κ-μ-g Random Variables

Desired signal in wireless communication systems can be subjected to cochannel interference (CCI) due to reused of radio frequencies which is main in increase the system capacity. Beside of fading, the wireless transmission can be degraded by shadowing which is the result of large obstacles and deviations in terrain profile between transmitter and receiver. In this composite fading-shadowing environment, the signal envelope can be modeled by product of two random variables [18]. Also, when two fading affect together at the combiner inputs, the equivalent envelope is equal to the product of two random variables.

The probability density function of product of two κ-μ-g random variables x_1 and x_2 is:

$$p_x(x) = \int_0^\infty dx_2 \frac{1}{x_2} p_{x_1}\left(\frac{\eta}{x_2}\right) p_{x_2}(x_2) =$$

$$= \int_0^\infty dx_2 \frac{1}{x_2} \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_1=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot$$

$$\cdot \left(\frac{\eta}{x_2}\right)^{2i_1+2\mu-1} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot \int_0^\infty d\Omega_1\, \Omega_1^{c-1-i_1-\mu} e^{-\frac{\mu(k+1)}{\Omega_1}\frac{\eta^2}{x_2^2}-\frac{1}{\beta}\Omega_1} \cdot$$

$$\cdot \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_2=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_2+\mu-1} \frac{1}{i_2!\Gamma(i_2+\mu)} \cdot$$

$$\cdot x_2^{2i_1+2\mu-1} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot \int_0^\infty d\Omega_2\, \Omega_2^{c-1-i_2-\mu} e^{-\frac{\mu(k+1)}{\Omega_2}x_2^2-\frac{1}{\beta}\Omega_2} =$$

$$= \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_1=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot$$

$$\cdot \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_2=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_2+\mu-1} \frac{1}{i_2!\Gamma(i_2+\mu)} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \eta^{2i_1+2\mu-1} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot \int_0^\infty d\Omega_1\, \Omega_1^{c-1-i_1-\mu} e^{-\frac{1}{\beta}\Omega_1} \cdot \int_0^\infty d\Omega_2\, \Omega_2^{c-1-i_2-\mu} e^{-\frac{1}{\beta}\Omega_2} \cdot$$

$$\cdot \int_0^\infty dx_2 \cdot x_2^{-1-2i_1-2\mu+1+2i_2+2\mu-1} \cdot e^{-\frac{\mu(k+1)}{\Omega_1}\frac{\eta^2}{x_2^2}-\frac{\mu(k+1)}{\Omega_2}x_2^2} =$$

$$= \left(\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}}\right)^2 \cdot \sum_{i_1=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot$$

$$\cdot \sum_{i_2=0}^\infty \left(\mu\sqrt{k(k+1)}\right)^{2i_2+\mu-1} \frac{1}{i_2!\Gamma(i_2+\mu)} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \frac{1}{\Gamma^2(c)} \cdot \frac{1}{\beta^{2c}} \eta_1^{2i_1+2\mu-1} \cdot \int_0^\infty d\Omega_1\, \Omega_1^{c-1-i_1-\mu} e^{-\frac{1}{\beta}\Omega_1}$$

$$\int_0^\infty d\Omega_2\, \Omega_2^{c-1-i_2-\mu} e^{-\frac{1}{\beta}\Omega_2}$$

$$\cdot \left(\frac{\Omega_2 y^2}{\Omega_1}\right)^{-\frac{i_2}{2}-\frac{i_1}{2}} \cdot K_{i_2-1}\left(2\sqrt{\frac{\mu^2(k+1)^2 y^2}{\Omega_1\Omega_2}}\right) \tag{22}$$

3 Numerical Results

The maximum of two κ-μ-g random variables is shown in Figs. 1 and 2. The maximum of two κ-μ-g random variables depending on Rician factor κ_1 for $c_1=c_2=\mu_1=\mu_2=2$ and variable parameters κ_2, β_1 and β_2 is presented in Fig. 1.

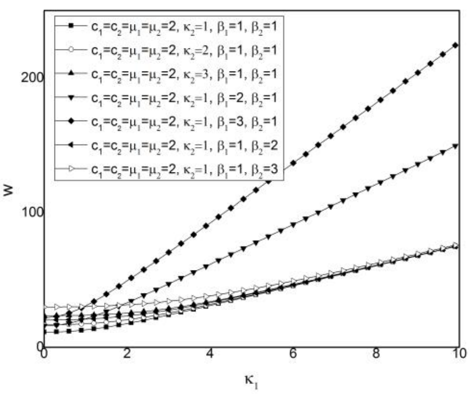

Fig.1. The maximum of two κ-μ-g random variables versus Rician factor κ_1.

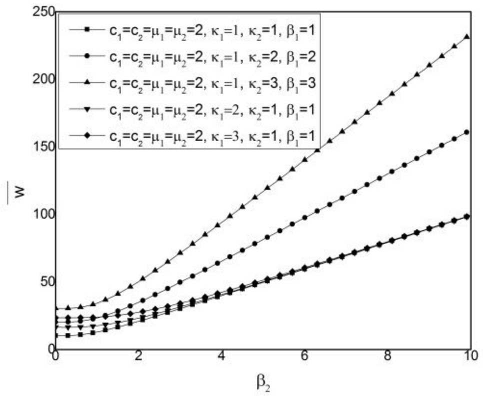

Fig. 2. The main value of the maximum of two κ-μ-g random variables depending of the parameter β_2.

The maximum of two κ-μ-g random variables versus parameter β_2, for $c_1 = c_2 = \mu_1 = \mu_2 = 2$ and variable Rician parameters κ_1, κ_2 and parameter β_1 is plotted in Fig. 2.

One can see from these two figures that maximum of two κ-μ-g random variables rises with growth of Rician factor κ_1.The maximum is greater for bigger parameter β_2. Also, the maximum is higher for larger parameters β_1 and κ_2.

The minimum of two κ-μ-g random variables is presented in Figs. 3 to 6. The mean value of minimum of two κ-μ-g random variables versus Rician factor κ_1 for $c_1=c_2 =\mu_1= \mu_2=2$, $\kappa_2=1$ and variable parameters β_1 and β_2 is shown in Fig. 3. The minimum of two κ-μ-g random variables versus Rician factor κ_2 for $c_1=c_2=\mu_1=\mu_2=2$ and changeable parameters κ_1, β_1 and β_2 is given in Fig. 4.

One can see from these two that minimum increases with enlarging of Rician factors $\kappa_1$and κ_2, but when it reaches a maximum, it stays with it for all other values of the Rician factors.

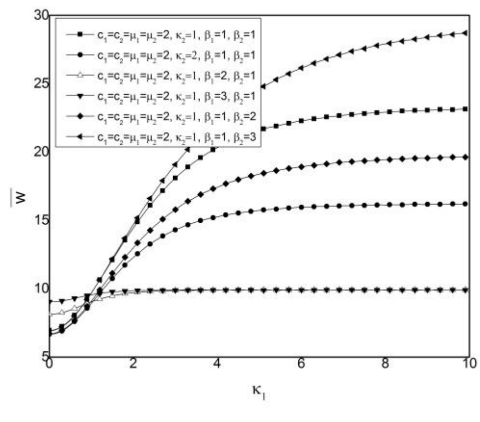

Fig. 3. Mean value of minimum of two κ-μ-g random variables depending on the Rician factor κ_1.

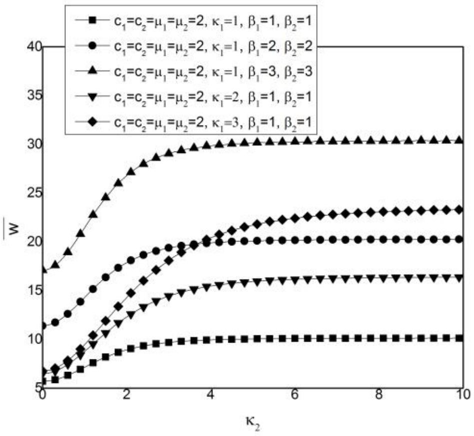

Fig. 4. Mean value of the minimum of two κ-μ-g random variables versus Rician factor κ_2.

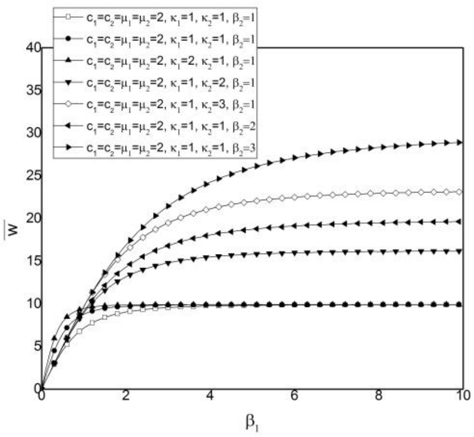

Fig. 5. Mean value of the minimum of two κ-μ-g random variables depending on the parameter β_1.

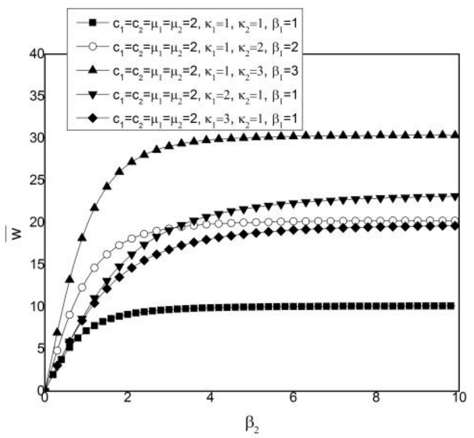

Fig. 6. Mean value of the minimum of two κ-μ-g random variables versus parameter β_2.

Mean value of the minimum of two κ-μ-g random variables depending on the parameter β_1 for $c_1=c_2=\mu_1=\mu_2=2$ and variable parameters κ_1, κ_2, β_1 and β_2 is drawn in Fig. 5. In the Fig. 6, the minimum of two κ-μ-g random variables versus parameter β_2 is presented for $c_1=c_2=\mu_1=\mu_2=2$, and variable β_1 and Rician factors κ_1 and κ_2.

The dependence of minimum from parameters β_1 and β_2 is bigger for smaller values of these parameters. The curves achieve maximums and stay with these values for all rest values of parameters β_1 and β_2.

The mean values of the ratios of two κ-μ-g random variables versus different fading parameters are presented in Figs. 7 to 10 [2]. In Figs. 7 and 8, the mean values of the ratio of two κ-μ-g random variables versus Rician factors κ_1, i.e. κ_2, for $c_1=c_2=\mu_1=\mu_2=2$ and $\beta_1=\beta_2=1$, and variable other parameter κ, are plotted.

One can see from these few figures that change of parameter κ_2 has small influence to the mean values of the ratio of two κ-μ-g random variables. The increasing of κ_2 leads to decreasing of the mean values of the ratio of two κ-μ-g random variables. On the other side, an increase of parameter κ_1 brings to increase of the mean values of this ratio.

In Figs. 9 and 10, the mean values of the ratio of two κ-μ-g random variables versus parameters β are given. It is visible from these figures that mean value of the ratio of two κ-μ-g random variables growing when parameter β_1 increases, and becomes smaller when parameter β_2 is higher.

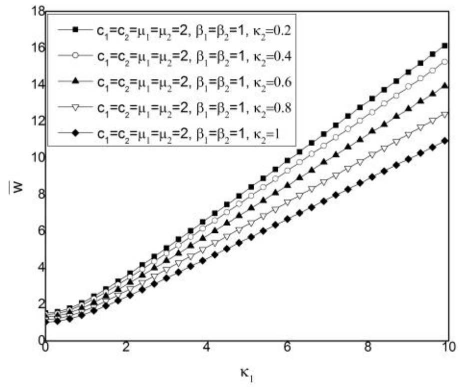

Fig. 7. Mean value of ratio of two κ-μ-g random variables depending on the Rician factor κ_1.

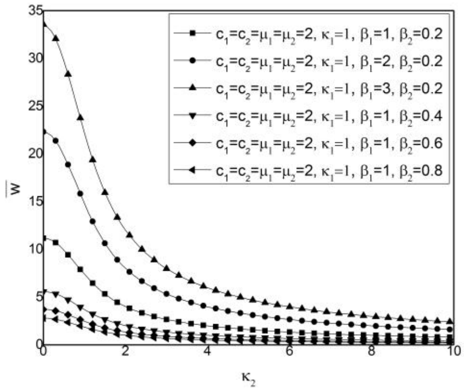

Fig. 8. First moment of the ratio of two κ-μ-g random variables versus Rician factor κ_2.

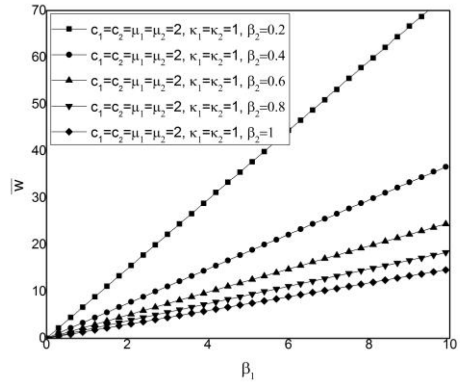

Fig. 9. Mean value of ratio of two κ-μ-g random variables depending on the parameter β_1.

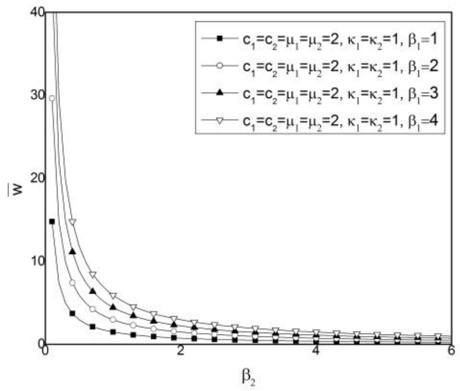

Fig. 10. Mean value of ratio of two κ-μ-g random variables versus parameter β_2.

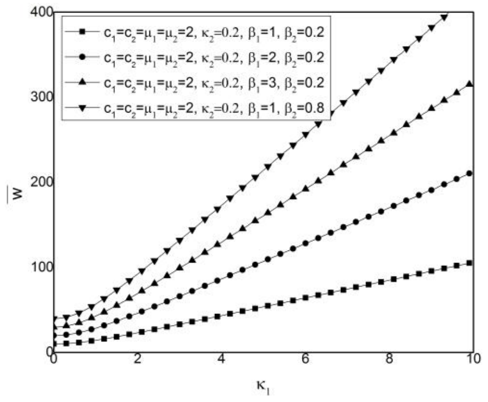

Fig. 11. Mean value of product of two κ-μ-g random variables versus Rician factor κ_1.

The mean values of the product of two κ-μ-g random variables versus fading parameters are plotted in Figs. 11. to 14 [2]. The mean values of product of two κ-μ-g random variables versus Rician factors κ_1 for $c_1 = c_2 = \mu_1 = \mu_2 = 2$, $\kappa_2 = 0.2$ and variable parameters β_1 and β_2 is presented in Fig. 11. The mean values of product of two κ-μ-g random variables depending of Rician factors κ_2 is given in Fig. 12. The mean values of the product of two κ-μ-g random variables versus parameters β_1 and β_2 are shown in Figs. 13. and 14. First, in Fig. 13, the graphs are shown for: $\beta_2=0.2$ and variable Rician factors κ_1 and κ_2, while the parameters of curves in Fig. 14 are: $\beta_1=1$, Rician factors κ_1 and κ_2 are changable, and $c_1=c_2=\mu_1=\mu_2=2$ is valide in both figures.

It can be seen from these figures that increasing of parameters $\kappa_1$and κ_2 leads to the increase of the mean values of the product of two κ-μ-g random variables.

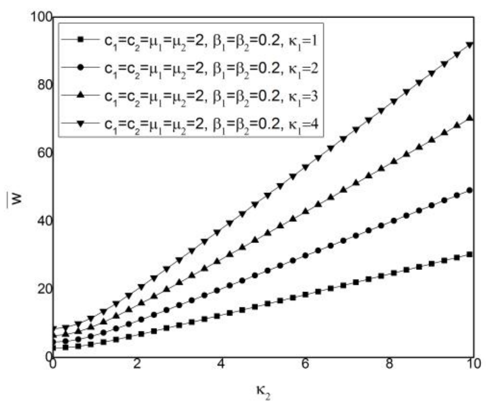

Fig. 12. Mean value of product of two κ-μ-g random variables depending on the Rician factor κ_2.

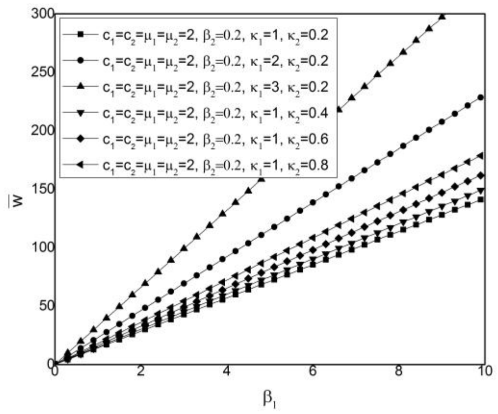

Fig. 13. Mean value of product of two κ-μ-g random variables depending on parameter β_1.

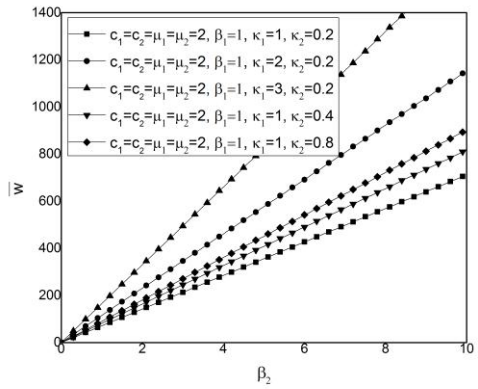

Fig. 14. Mean value of the product of two κ-μ-g random variables depending on the parameter β_2.

Especially, from Figs. 13 and 14, one can notice that bigger parameters β_1 and β_2 has bigger influence to enlarging of the mean value of product of two κ-μ-g random variables. With the increase of the mean value of useful signal, the system performances are improving. This can be accomplished with an increasing of the power of the dominant component and reducing the power of scattering components. With growing of the power of scattering components, the outage probability is getting bigger, what worsens the system performance.

It is obvious from Figs. 7. to 10. that an increase of parameters κ_1 and β_1 leads to an increase of the mean value of the signals ratio, reducing the error probability and improvement of system performance. When parameters κ_2 and β_2 are increasing, the signal mean value decreases and the error probability increases. From Figs. 11 to 14 is visible that system performance is better for bigger values of parameters κ1 and κ2 and smaller values of parameters β1 and β2.

The designers of wireless telecommunications systems can use this analysis to select the optimal system parameters, for which the mean value of the signal is growing. Then, the system error probability decreases and performance of wireless communication system is better.

4 Conclusion

The κ-μ-g random variable is discussed in this paper. The κ-μ-g random variable arises from the κ-μ random variable with Gamma distributed power of κ-μ random process.

The closed form expression for probability density function and cumulative distribution function of κ-μ-g random variable are determined. The obtained expressions can be used in performance analysis of wireless communication systems operating over κ-μ multipath fading channels undergo Gamma shadowing.

In this paper, the maximum of two κ-μ-g random variables is processed. PDF and CDF of the maximum of two κ-μ-g random variables are derived. Statistics of the maximum of two κ-μ-g random variables can be used in performance analysis of wireless communication systems with SC combiner with two branches in the presence of Gamma shadowed κ-μ multipath fading.

The statistics of minimum of two random variables is also analyzed. It is necessary for performance evaluation of wireless relay communication systems with two sections. Under determined conditions, signal envelope at output of relay system can be expressed as product of signal envelope at each section. Cumulative distribution of minimum of two random variables is used for calculation the outage probability of relay system with two sections.

The distribution of the ratio of random variables is also investigated because it is important in statistical analysis in wireless communication systems in the presence of fading. In this case, the random variable in nominator presents the desired signal envelope while the random variable in denominator presents the interference signal envelope. This is important in environment where interference is present during transmition.

In composite shadowed fading environment, the signal envelope is modeled by product of two random variables. In the situations when two fading affect together at the combiner inputs, the equivalent envelope is equal to the product of two random variables. Because of that, investigation of PDF of product is very important.

At the last section, the influence of Rician factors and Gamma long term fading severity parameters on the performance of two κ-μ-g random variables is analyzed. This analysis is motivated by the fact that this distribution evinces an excellent agreement to experimental fading channel conditions. An application of these results for the wireless communications community is very important.

In order to provide enough information for the overall system design and configuration, deriving expressions for moments-generating function will be the subject of our future work.

Appendix

We consider and solve the integral [2]:

$$\int_0^\infty ds\, s^{p_1-1} e^{-\alpha_1 s} \int_0^\infty d\Omega\, \Omega^{p_2-1} e^{-\alpha_2\Omega} \frac{1}{(a\Omega+bs)^n} =$$

$$= \frac{1}{\alpha_2^{p_1+p_{w22}-n} b^{p_1}} a^{p_1-n}\Gamma(p_2)$$

$$\frac{\Gamma(p_1+p_2-n)\Gamma(p_1)}{\Gamma(p_1+p_2)}$$

$${}_2F_1\left(p_1+p_2-n, p_1, p_1+p_2; 1-\frac{\alpha_1 a}{\alpha_2 b}\right)$$

For considered case, the parameters are:

$$p_1 = c_1 + i_2 + \mu_2$$

$$p_2 = c_2 + i_1 + \mu_1$$

$$\alpha_1 = \frac{1}{\beta_1}$$

$$\alpha_2 = \frac{1}{\beta_2}$$

$$b = \mu_1(\kappa_1+1)$$

$$a = \mu_2(\kappa_2+1)$$

$$n = i_1 + \mu_1 + i_2 + \mu_2$$

$$p_1 + p_2 - n =$$

$$= c_1 + c_2 + i_1 + i_2 + \mu_1 + \mu_2 - i_1 - i_2 - \mu_1 - \mu_2 =$$

$$= c_1 + c_2$$

$$p_1 + p_2 = c_1 + c_2 + i_1 + i_2 + \mu_1 + \mu_2$$

$$p_1 - n = c_1 + i_2 + \mu_2 - i_1 - i_2 - \mu_1 - \mu_2 =$$

$$= c_1 - i_1 - \mu_1$$

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