# A Structural Approach to Model Following by Output Feedback in Hybrid Linear Systems with State Jumps

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*Abstract:* The topic of this work is model following by output feedback in hybrid linear systems with state jumps. In particular, the hybrid linear systems addressed are allowed to exhibit a direct algebraic link from the control input to the regulated output — henceforth referred to as the control feedthrough. Moreover, this study considers the problem of model following from a structural point of view in the sense that it is focused on achieving that the output of the compensated system perfectly follows that of the model, provided that both the compensated system and the model have zero initial conditions, for all the admissible input signals. A necessary and sufficient condition for the existence of a solution to the problem dealt with is proven.

*Key–Words:* Hybrid systems, Model following, Structural approach

## 1 Introduction

Model following is a widely studied problem of system and control theory. First stated and solved for linear time-invariant systems by state feedback [1], it was lately solved by means of other compensation schemes still in the framework of linear systems [2,3]. Moreover, it was extended to other classes of dynamical systems, such as nonlinear systems [4, 5], time-delay and uncertain systems [6–8], large-scale systems [9, 10], Markovian jump linear systems [11], multimodal switching systems [12–16]. The interest of model following is also related to the number of applications which have been developed during the last decades [17–28].

In this work, model following is studied for a special class of hybrid dynamical systems — namely, those characterized by a continuous-time linear behavior subject to abrupt state discontinuities. The dynamics ruling the continuous-time behavior is briefly called *flow dynamics*, while the dynamics governing the instantaneous changes of the state is called *jump dynamics*. More precisely, this work is focused on hybrid linear systems whose jumps are possibly non-equally spaced in time, but satisfy the only constraint that the number of jump times is finite in any finite time interval, so as to leave possible chattering phenomena out of consideration. Moreover, the hybrid linear systems addressed exhibit a direct control feedthrough on the to-be-controlled output and allow a control input on the sole flow dynamics.

Hybrid linear systems with state jumps, in general, have recently focused the attention of the scientific community, mainly for their capability of capturing the features of complex dynamical systems, such as colliding mechanical systems, multi-agent systems, electro-mechanical systems and many others — see, e.g., [29]. For this reason, several control problems have been formalized and investigated for these dynamical systems in some previous papers [30–32] and the aim of this work is to extend the previously developed methodologies so as to handle model following. In particular, the possible presence of the control feedthrough requires to elaborate further on recently-introduced geometric notions for hybrid systems.

The methodology devised in this work in order to handle hybrid linear systems is grounded on the geometric approach to linear control theory [33, 34]. Although the geometric approach is a well-settled methodology, introduced in the late sixties, it has recently proven to be very flexible and powerful in dealing with various control problems and different kind of dynamical systems [35–43]. In particular, this approach has been applied to the study of other classes of hybrid systems, such as switching multimodal systems [44–54].

### 1.1 Notation

The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}^+$ stand for the sets of real numbers, nonnegative real numbers, and nonnegative integer numbers, respectively. Matrices and linear maps are denoted by slanted upper-case letters, like $A$. The image and the kernel of $A$ are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The transpose of $A$ is denoted by $A^\top$. The inverse of a nonsingular square matrix $A$ is denoted by $A^{-1}$. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The symbol $I$ denotes an identity matrix of appropriate dimension.

## 2 Model Following by Output Feedback in Hybrid Linear Systems: Problem Statement

In order to define a hybrid linear systems with state jumps, the hybrid time domain must be set first. The symbol $\mathcal{T}$ denotes a finite or countably infinite ordered set $\{t_0, t_1, \ldots\}$ of strictly increasing elements of $\mathbb{R}^+$. The symbol $t_f$ stands for the last element of $\mathcal{T}$ when $\mathcal{T}$ has finite cardinality. The set $\mathcal{T}$ is assumed to exhibit no accumulation points. The symbol $\mathscr{T}$ denotes the set of all $\mathcal{T}$ meeting the constraint mentioned above. The nonnegative real axis deprived of the elements of $\mathcal{T}$ is denoted by $\mathbb{R}^+ \setminus \mathcal{T}$.

The hybrid linear system with state jumps $\Sigma_P$ is defined by

$$\Sigma_P \equiv \begin{cases} \dot{x}_P(t) &= A_P\, x_P(t) + B_P\, u(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_P(t_k) &= G_P\, x_P(t_k^-), \quad t_k \in \mathcal{T}, \\ e_P(t) &= E_P\, x_P(t) + D_P\, u(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where $x_P \in \mathcal{X}_P = \mathbb{R}^{n_P}$ is the state, $u \in \mathbb{R}^p$ is the control input, and $e_P \in \mathbb{R}^q$ is the output, with $p, q \leq n_P$. $A_P$, $B_P$, $G_P$, $E_P$, and $D_P$ are constant real matrices of appropriate dimensions. The algebraic link from the control input to the output, established by the matrix $D_P$, is referred to as the *control feedthrough*. The rank of the matrices

$$\begin{bmatrix} B_P \\ D_P \end{bmatrix}, \quad \begin{bmatrix} E_P & D_P \end{bmatrix},$$

is assumed to be full. The set of the admissible control input functions $u(t)$, with $t \in \mathbb{R}^+$, is defined as the set of all piecewise-continuous functions with values in $\mathbb{R}^p$. The so-called *flow dynamics* is ruled by the differential state equation. Meanwhile, the algebraic state equation governes the so-called *jump dynamics*. Thus, according to the hybrid linear structure of $\Sigma_P$, the state motion $x_P(t)$ in $[0,\, t_0)$ is the solution of the differential equation, with given initial state $x_P(0) = x_{S,0}$ and input function $u(t)$, with $t \in [0,\, t_0)$. The state $x_P(t_k)$, with $t_k \in \mathcal{T}$, is the image through $G_P$ of $x_P(t_k^-) = \lim_{\epsilon \to 0^+} x_P(t_k - \epsilon)$. The state motion $x_P(t)$ in $[t_k,\, t_{k+1})$, with $t_k, t_{k+1} \in \mathcal{T}$, is the solution of the differential equation, given the initial state $x_P(t_k)$ and the input function $u(t)$, with $t \in [t_k,\, t_{k+1})$.

The hybrid linear reference model with state jumps $\Sigma_R$ is defined by

$$\Sigma_R \equiv \begin{cases} \dot{x}_R(t) &= A_R\, x_R(t) + B_R\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_R(t_k) &= G_R\, x_R(t_k^-), \quad t_k \in \mathcal{T}, \\ e_R(t) &= E_R\, x_R(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where $x_R \in \mathbb{R}^{n_R}$ is the state, $d \in \mathbb{R}^q$ is the input, and $e_R \in \mathbb{R}^q$ is the output. The set of the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$, is defined as the set of all piecewise-continuous functions with values in $\mathbb{R}^q$.

Hence, the problem of model following by output feedback in hybrid linear systems with state jumps is cast as follows. A block diagram illustrating the system interconnection referred to in Problem 1 below is presented in Figure 1.

**Problem 1 (Model Following by Output Feedback in Hybrid Linear Systems with State Jumps)** *Let the hybrid linear system with state jumps $\Sigma_P$ and the hybrid linear reference model with state jumps $\Sigma_R$ be given. Find a hybrid linear compensator with state jumps $\Sigma_C$, defined by*

$$\Sigma_C \equiv \begin{cases} \dot{x}_C(t) &= A_C\, x_C(t) + B_C\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_C(t_k) &= G_C\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ u(t) &= C_C\, x_C(t), \quad t \in \mathbb{R}^+, \end{cases}$$

*where $h(t) = d(t) - e_P(t)$, such that the closed-loop hybrid linear system with state jumps $\Sigma_O$, defined by*

$$\Sigma_O \equiv \begin{cases} \dot{x}_O(t) &= A_O\, x_O(t) + D_O\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_O(t_k) &= G_O\, x_O(t_k^-), \quad t_k \in \mathcal{T}, \\ e_P(t) &= E_O\, x_O(t), \quad t \in \mathbb{R}^+, \end{cases}$$

*where*

$$A_O = \begin{bmatrix} A_P & B_P\, C_C \\ -B_C\, E_P & A_C - B_C\, D_P\, C_C \end{bmatrix},$$

$$D_O = \begin{bmatrix} 0 \\ B_C \end{bmatrix},$$

$$G_O = \begin{bmatrix} G_P & 0 \\ 0 & G_C \end{bmatrix},$$

$$E_O = \begin{bmatrix} E_P & D_P\, C_C \end{bmatrix},$$

*satisfies the requirement that the output $e_P(t)$ is equal to the reference model output $e_R(t)$, for all $t \in \mathbb{R}^+$, when the respective initial states are zero, for all the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$, and all the admissible sequences of jump times $\mathcal{T} \in \mathscr{T}$.*

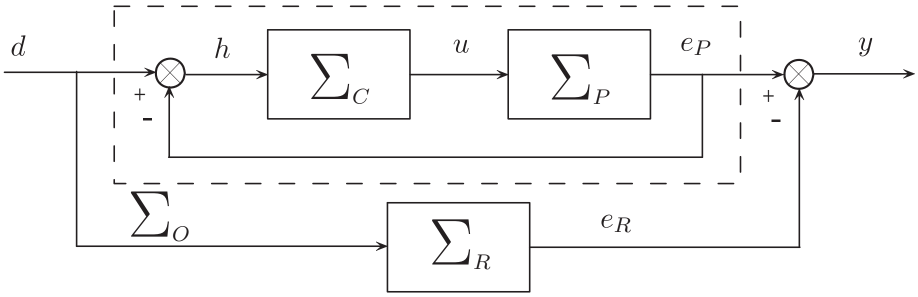


Figure 1: Model Following by Output Feedback

## 3 Feedforward Disturbance Decoupling for the Extended Hybrid Linear System: Problem Statement

As will be shown later on, the solution to the problem stated in Section 2 can be achieved by solving the problem which is the object of this section: namely, a problem of disturbance decoupling by dynamic feedforward, stated for a suitably-defined hybrid linear system with state jumps. This newly-defined hybrid linear system — from now on called the *extended* hybrid linear system with state jumps — is the output-difference connection between the given hybrid linear plant $\Sigma_P$ and a modified hybrid linear reference model — henceforth denoted by $\Sigma_R^+$. In particular, the hybrid linear reference model $\Sigma_R^+$ is derived from the original model $\Sigma_R$ by closing a positive unit feedback of the output on the flow dynamics. Thus, $\Sigma_R^+$ is ruled by

$$\Sigma_R^+ \equiv \left\{ \begin{aligned} \dot{x}_R(t) &= (A_R + B_R\,E_R)\,x_R(t) + B_R\,h(t), \\ &\qquad\qquad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_R(t_k) &= G_R\,x_R(t_k^-), \quad t_k \in \mathcal{T}, \\ e_R(t) &= E_R\,x_R(t), \quad t \in \mathbb{R}^+. \end{aligned} \right.$$

The set of the admissible input functions to the modified reference model $\Sigma_R^+$ is defined as the set of all piecewise-continuous functions $h(t)$, with $t \in \mathbb{R}^+$, picking their values in $\mathbb{R}^q$.

Consequently, the extended hybrid linear system with state jumps — denoted by $\Sigma$ — is defined as the connection of the given hybrid linear systems $\Sigma_P$ with the modified hybrid linear reference model $\Sigma_R^+$, such that the control input, the disturbance input, and the output of $\Sigma$ respectively are the control input of $\Sigma_P$, the input of $\Sigma_R^+$, and the difference between the outputs of $\Sigma_P$ and $\Sigma_R^+$. Therefore, $\Sigma$ is described by

$$\Sigma \equiv \left\{ \begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + H\,h(t), \ t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x(t_k) &= G\,x(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) &= E\,x(t) + D\,u(t), \quad t \in \mathbb{R}^+, \end{aligned} \right.$$

where

$$A = \begin{bmatrix} A_P & 0 \\ 0 & A_R + B_R\,E_R \end{bmatrix},$$
$$B = \begin{bmatrix} B_P \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ B_R \end{bmatrix},$$
$$G = \begin{bmatrix} G_P & 0 \\ 0 & G_R \end{bmatrix},$$
$$E = \begin{bmatrix} E_P & -E_R \end{bmatrix}, \quad D = D_P.$$

The state space of $\Sigma$ will be denoted by $\mathcal{X}$: i.e., $\mathcal{X} = \mathbb{R}^n$, where $n = n_P + n_R$.

Hence, the disturbance decoupling problem by dynamic feedforward, for the extended hybrid linear systems with state jumps $\Sigma$ can be stated as follows. Figure 2 shows a block diagram of the system interconnection dealt with in Problem 2 below.

**Problem 2 (Feedforward Disturbance Decoupling for the Extended Hybrid Linear System with State Jumps)** *Let the extended hybrid linear system with state jumps $\Sigma$ be given. Find a hybrid linear compensator with state jumps $\Sigma_C$ such that the compensated hybrid linear systems*

$$\bar{\Sigma} \equiv \left\{ \begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\,\bar{x}(t) + \bar{H}\,h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \bar{x}(t_k) &= \bar{G}\,\bar{x}(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) &= \bar{C}\,\bar{x}(t), \quad t \in \mathbb{R}^+, \end{aligned} \right.$$

*where*

$$\bar{A} = \begin{bmatrix} A & B\,C_C \\ 0 & A_C \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H \\ B_C \end{bmatrix},$$
$$\bar{G} = \begin{bmatrix} G & 0 \\ 0 & G_C \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} E & D\,C_C \end{bmatrix},$$

*satisfies the requirement that the output $e(t)$ is zero, for all $t \in \mathbb{R}^+$, when the initial state is zero, for all the admissible input functions $h(t)$, with $t \in \mathbb{R}^+$, and all the admissible sequences of jump times $\mathcal{T} \in \mathscr{T}$.*

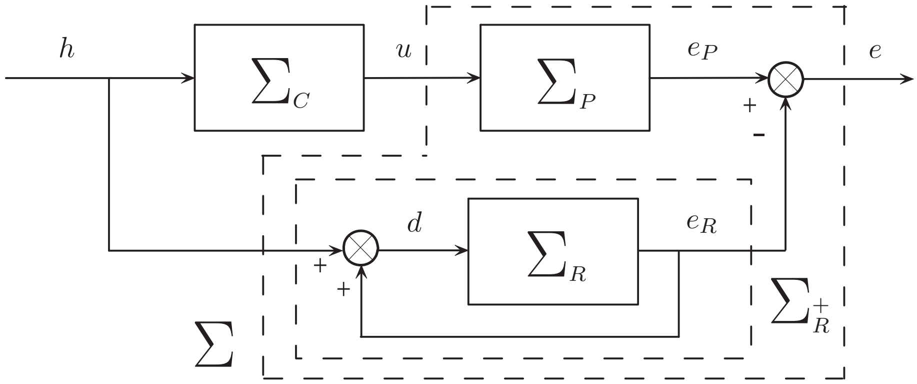


Figure 2: Feedforward Disturbance Decoupling

## 4 Feedforward Disturbance Decoupling for the Extended Hybrid Linear System: Problem Solution

Solvability of Problem 2 can be completely characterized by a necessary and sufficient condition exploiting the geometric notions introduced in the Appendix. As will be shown in this section, such condition can be expressed in coordinate-free terms, since it amounts to an inclusion of subspaces. Nevertheless, since the proof of sufficiency is constructive — namely, it is includes the synthesis of the compensator — some preliminary remarks are also made with the purpose of expressing such condition with reference to suitably chosen coordinates.

To begin with, it is worth highlighting that the linear map $A + B\,K$, where $K$ is a friend of the maximal output-nulling $\mathscr{H}$-controlled invariant subspace $\mathcal{W}^*_{\mathscr{H}}$, is represented by a matrix with a typical upper block-triangular structure, when a suitable similarity transformation is applied to the state space. In particular, let $S$ be a change of basis defined by $S = [\,S_1\;\, S_2\,]$, with $\mathrm{Im}\,S_1 = \mathcal{W}^*_{\mathscr{H}}$. Then, in the new coordinates,

$$A' + B'\,K' = S^{-1}\,(A + B\,K)\,S = \begin{bmatrix} A'_{11} + B'_1\,K'_1 & A'_{12} + B'_1\,K'_2 \\ 0 & A'_{22} + B'_2\,K'_2 \end{bmatrix}, \qquad (1)$$

where the structural zero in the lower left corner — i.e.,

$$A'_{21} + B'_2\,K'_1 = 0, \qquad (2)$$

is due to $(A + B\,K)$-invariance of $\mathcal{W}^*_{\mathscr{H}}$. Similarly, the linear map $G$ is represented by

$$G' = S^{-1}\,G\,S = \begin{bmatrix} G'_{11} & G'_{12} \\ 0 & G'_{22} \end{bmatrix}, \qquad (3)$$

in the same coordinates, where the structural zero in the lower left corner is due to $G$-invariance of $\mathcal{W}^*_{\mathscr{H}}$. Moreover, with respect to the same coordinates, the linear map $E + D\,K$, where $K$ is the friend of $\mathcal{W}^*_{\mathscr{H}}$ considered, is represented by a matrix with a structural zero in the first block of columns: i.e.,

$$\begin{aligned} E' + D'\,K' &= (E + D\,K)\,S \\ &= \begin{bmatrix} 0 & E'_2 + D\,K'_2 \end{bmatrix}, \end{aligned} \qquad (4)$$

where the structural zero — i.e.,

$$E'_1 + D\,K'_1 = 0, \qquad (5)$$

is due to $\mathcal{W}^*_{\mathscr{H}} \subseteq \mathrm{Ker}\,(E + D\,K)$.

Furthermore, the subspace inclusion that will be proven to be equivalent to solvability of Problem 2 can be conveniently recast in a coordinate-dependent form with reference to the basis considered above. This is to say that

$$\mathcal{H} \subseteq \mathcal{W}^*_{\mathscr{H}} \qquad (6)$$

is equivalent to

$$H' = S^{-1}\,H = \begin{bmatrix} H'_1 \\ 0 \end{bmatrix}. \qquad (7)$$

In fact, the structural zero in $H'$ means that a basis matrix of $\mathcal{H}$ is a linear combination of the column vectors of the basis matrix $S_1$ of $\mathcal{W}^*_{\mathscr{H}}$.

With these premises, the necessary and sufficient condition for Problem 2 to be solvable is formulated as in the following theorem.

**Theorem 3** *Let the hybrid linear system with state jumps $\Sigma$ be given. Problem 2 is solvable if and only if (6) holds.*

**Proof:** If. Let (6) hold. Let $K$ be a friend of $\mathcal{W}^*_{\mathscr{H}}$. Hence, (1), (3), (4), and (7) hold with respect to the specified coordinates. Let $A'_C = A'_{11} + B'_1\,K'_1$, $B'_C = H'_1$, $G'_C = G'_{11}$, and $C'_C = K'_1$ be the matrices of the hybrid linear regulator $\Sigma_C$ with respect to such

$$\bar{\Sigma} \equiv \begin{cases} \dot{x}_1(t) = A'_{11}\, x_1(t) + A'_{12}\, x_2(t) + B'_1\, K'_1\, x_C(t) + H'_1\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_2(t) = A'_{21}\, x_1(t) + A'_{22}\, x_2(t) + B'_2\, K'_1\, x_C(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_C(t) = (A'_{11} + B'_1\, K'_1)\, x_C(t) + H'_1\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_1(t_k) = G'_{11}\, x_1(t_k^-) + G'_{12}\, x_2(t_k^-), \quad t_k \in \mathcal{T}, \\ x_2(t_k) = G'_{22}\, x_2(t_k^-), \quad t_k \in \mathcal{T}, \\ x_C(t_k) = G'_{11}\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) = E'_1\, x_1(t) + E'_2\, x_2(t) + D\, K'_1\, x_C(t), \quad t \in \mathbb{R}^+, \end{cases} \tag{8}$$

$$\bar{\Sigma} \equiv \begin{cases} \dot{\zeta}(t) = A'_{11}\, \zeta(t) + A'_{12}\, x_2(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_2(t) = A'_{21}\, \zeta(t) + A'_{22}\, x_2(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_C(t) = (A'_{11} + B'_1\, K'_1)\, x_C(t) + H'_1\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \zeta(t_k) = G'_{11}\, \zeta(t_k^-) + G'_{12}\, x_2(t_k^-), \quad t_k \in \mathcal{T}, \\ x_2(t_k) = G'_{22}\, x_2(t_k^-), \quad t_k \in \mathcal{T}, \\ x_C(t_k) = G'_{11}\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) = E'_1\, \zeta(t) + E'_2\, x_2(t), \quad t \in \mathbb{R}^+, \end{cases} \tag{9}$$

coordinates. Then, it will be shown that $\Sigma_C$, with zero initial state, solves Problem 2. To this purpose, it is worth observing that the cascade, denoted by $\bar{\Sigma}$ in Problem 2, of the hybrid linear compensator $\Sigma_C$ (thus determined) with the extended hybrid linear system $\Sigma$ is ruled by (8), where the state of $\Sigma$, in the new coordinates, is partitioned as $\left[x_1^\top \; x_2^\top\right]^\top$ according to (1), (3), (4), and (7). By setting $\zeta(t) = x_1(t) - x_C(t)$, with $t \in \mathbb{R}^+$, the system $\bar{\Sigma}$ can be recast as in (9), where (2) and (5) have been taken into account. Hence, the assumption that the initial state is zero implies $\zeta(t) = 0$ and $x_2(t) = 0$, for all $t \in \mathbb{R}^+$, which also implies $e(t) = 0$, for all $t \in \mathbb{R}^+$, for all the admissible input functions $h(t)$, with $t \in \mathbb{R}^+$, and all the admissible jump time sequences $\mathcal{T} \in \mathscr{T}$.

Only if. If (6) does not hold, no other output-nulling $\mathscr{H}$-controlled invariant subspace containing $\mathcal{H}$ exists, since the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is an upper semilattice and $\mathcal{W}^*_{\mathscr{H}}$ is the maximum. □

# 5 Model Following by Output Feedback in Hybrid Linear Systems: Problem Solution

This section is aimed at showing that the problem of disturbance decoupling by dynamic feedforward stated and solved for the extended hybrid linear system with state jumps respectively in Sections 3 and 4 is equivalent to the problem of model following by output feedback stated in Section 2. This fact will be proven by demonstrating that a hybrid linear compensator with state jumps solves one of these problems if and only if it solves the other. This result is formalized in the theorem below.

**Theorem 4** *A hybrid linear compensator with state jumps $\Sigma_C$ solves Problem 2 if and only if it solves Problem 1.*

**Proof:** If. Let the hybrid linear compensator $\Sigma_C$ solve Problem 1. Consequently, the overall hybrid linear system with output feedback — from now called $\bar{\Sigma}'$ — is ruled by (10). It is worthwhile observing that, since $\Sigma_C$ solves Problem 1, under the assumption that the initial state is the origina, the output of $\bar{\Sigma}'$ satisfies the condition that $e(t) = 0$, for all $t \in \mathbb{R}^+$, for all the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$. Therefore, one can replace $e_P(t) = E_P\, x_P(t) + D_P\, C_C\, x_C(t)$ with $e_R(t) = E_R\, x_R(t)$ in the state equations of $\bar{\Sigma}'$. Consequently, the new system $\bar{\Sigma}''$ is described by (11). Further, since $e(t) = 0$ for all $t \in \mathbb{R}^+$, for all the admissible $d(t)$, with $t \in \mathbb{R}^+$, such condition holds when $d(t) = h(t) + E_R\, x_R(t)$, where $h(t)$, with $t \in \mathbb{R}^+$, stands for any admissible input function. Then, the system which turns out is the hybrid linear system $\bar{\Sigma}$ considered in Problem 2, as is proven by (12), which derive from $\bar{\Sigma}''$ with the abovementioned replacement. The equations of $\bar{\Sigma}$, which hold with $e(t) = 0$ for all $t \in \mathbb{R}^+$, for all the admissible $h(t)$, with $t \in \mathbb{R}^+$, show that the hybrid linear compensator $\Sigma_C$ also solves Problem 2: i.e., the problem of decoupling the signal $h(t)$, with $t \in \mathbb{R}^+$, in the extended hybrid linear system $\Sigma$, including the modified hybrid linear reference model $\Sigma_R^+$.

Only if. Let the hybrid linear compensator with state jumps $\Sigma_C$ solve Problem 2. Therefore, to show that $\Sigma_C$ also solves Problem 1, the reasoning presented in the if-part of the proof of can be pursued

$$\bar{\Sigma}' \equiv \begin{cases} \dot{x}_P(t) = A_P\, x_P(t) + B_P\, C_C\, x_C(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_C(t) = -B_C\, E_P\, x_P(t) + (A_C - B_C\, D_P\, C_C)\, x_C(t) + B_C\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_R(t) = A_R\, x_R(t) + B_R\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_P(t_k) = G_P\, x_P(t_k^-), \quad t_k \in \mathcal{T}, \\ x_C(t_k) = G_C\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ x_R(t_k) = G_R\, x_R(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) = E_P\, x_P(t) + D_P\, C_C\, x_C(t) - E_R\, x_R(t), \quad t \in \mathbb{R}^+. \end{cases} \tag{10}$$

$$\bar{\Sigma}'' \equiv \begin{cases} \dot{x}_P(t) = A_P\, x_P(t) + B_P\, C_C\, x_C(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_C(t) = A_C\, x_C(t) - B_C\, E_R\, x_R(t) + B_C\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_R(t) = A_R\, x_R(t) + B_R\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_P(t_k) = G_P\, x_P(t_k^-), \quad t_k \in \mathcal{T}, \\ x_C(t_k) = G_C\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ x_R(t_k) = G_R\, x_R(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) = E_P\, x_P(t) + D_P\, C_C\, x_C(t) - E_R\, x_R(t), \quad t \in \mathbb{R}^+. \end{cases} \tag{11}$$

backward — namely, starting from $\bar{\Sigma}$ and ending to $\bar{\Sigma}'$. □

## 6 Concluding Remarks

This work has been focused on hybrid linear systems with state jumps and has shown a necessary and sufficient condition to achieve structural model following between a given plant and a given reference model by means of an output feedback compensator. The theoretical result also outlines an algorithmic procedure for the synthesis of the hybrid linear compensator that solves the problem.

## A Appendix

The wide literature available on disturbance decoupling shows that one of the most powerful tools to successfully handle this problem is the geometric approach [33, 34]. Indeed, during the last decades, the fundamental concepts formerly established to deal with linear time-invariant systems have been extended to more complex dynamical systems. More specifically, concerning hybrid linear systems with state jumps, some basic ideas, such as invariance and controlled invariance, have been generalized so as to adapt to this kind of dynamical systems in some earlier articles [30–32]. However, in this work, as mentioned above, the considered hybrid linear systems may exhibit a direct algebraic link from the control input to the output. Hence, the notion of hybrid controlled invariance must be integrated by the novel extension of the idea of *output-nulling* controlled invariance.

The definitions of hybrid invariant subspace, hybrid controlled invariant subspace and output-nulling hybrid controlled invariant subspace are given below with reference to the extended hybrid linear system with state jumps $\Sigma$. Nonetheless, it is understood that the special structure of the matrices of $\Sigma$ does not affect those definitions. From now on, the symbol $\mathscr{H}$ stands to qualify *hybrid* invariance or, respectively, controlled invariance. The symbol $\mathcal{B}$ is the short notation for $\operatorname{Im} B$, while $\mathcal{H}$ stands for $\operatorname{Im} H$. A subspace $\mathcal{W} \subseteq \mathcal{X}$ is said to be an $\mathscr{H}$-invariant subspace if $A\,\mathcal{W} \subseteq \mathcal{W}$ and $G\,\mathcal{W} \subseteq \mathcal{W}$. A subspace $\mathcal{W} \subseteq \mathcal{X}$ is said to be an $\mathscr{H}$-controlled invariant subspace if $A\,\mathcal{W} \subseteq \mathcal{W} + \mathcal{B}$ and $G\,\mathcal{W} \subseteq \mathcal{W}$. Furthermore, it can be shown that a subspace $\mathcal{W} \subseteq \mathcal{X}$, with a basis matrix $W$, is an $\mathscr{H}$-controlled invariant subspace if and only there exist matrices $X_A$, $X_G$, and $U$ such that $A\,W = W\,X_A + B\,U$ and $G\,W = W\,X_G$. Hence, the definition of output-nulling $\mathscr{H}$-controlled invariant subspace is introduced as follows.

**Definition 5** *A subspace $\mathcal{W} \subseteq \mathcal{X}$, with a basis matrix $W$, is said to be an output-nulling $\mathscr{H}$-controlled invariant subspace if there exist matrices $X_A$, $X_G$, and $U$ such that $A\,W = W\,X_A + B\,U$, $G\,W = W\,X_G$, and $E\,W = D\,U$.*

A relevant characterization of the geometric concept of output-nulling $\mathscr{H}$-controlled invariant subspace is expressed by the following statement, whose proof directly steams from the properties enjoyed by simultaneous invariant and output-nulling controlled invariant subspaces in the linear time-invariant case.

**Proposition 6** *A subspace $\mathcal{W} \subseteq \mathcal{X}$ is an output-nulling $\mathscr{H}$-controlled invariant subspace if and only if there exists a linear map $K$ such that*

$$\bar{\Sigma} \equiv \begin{cases} \dot{x}_P(t) = A_P\, x_P(t) + B_P\, C_C\, x_C(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_C(t) = A_C\, x_C(t) + B_C\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_R(t) = (A_R + B_R\, E_R)\, x_R(t) + B_R\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_P(t_k) = G_P\, x_P(t_k^-), \quad t_k \in \mathcal{T}, \\ x_C(t_k) = G_C\, x_C(t_k^-), \quad t_k \in \mathcal{T}, \\ x_R(t_k) = G_R\, x_R(t_k^-), \quad t_k \in \mathcal{T}, \\ e(t) = E_P\, x_P(t) + D_P\, C_C\, x_C(t) - E_R\, x_R(t), \quad t \in \mathbb{R}^+. \end{cases} \tag{12}$$

*$(A + B\,K)\,\mathcal{W} \subseteq \mathcal{W}$ and $\mathcal{W} \subseteq \mathrm{Ker}\,(E + D\,K)$ hold along with $G\,\mathcal{W} \subseteq \mathcal{W}$.*

Any linear map $K$ satisfying the conditions of Proposition 6 is said to be a *friend* of the output-nulling $\mathscr{H}$-controlled invariant subspace $\mathcal{W}$.

As can be shown by simple algebraic arguments, the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is an upper semilattice with respect to the sum and the inclusion of subspaces. The maximum of the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is henceforth denoted by $\mathcal{W}^*_{\mathscr{H}}$.

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